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Solution 8

Randomized Algorithms and Probabilistic Methods: Advanced Topics

Solution to Exercise 1

For the one bound, we consider the cycle C_n on n vertices. In another exercise, we saw that the eigenvalues of C_n are $2\cos(2\pi k/n)$, where $k = 0, 1, \ldots, n-1$. Thus, for C_n (which is d-regular with $d = 2$), the spectral gap satisfies

$$d - \lambda_2 = 2 - 2\cos(2\pi/n) \le 2 - 2\Big(1 - \frac{(2\pi)^2}{2!n^2}\Big) = \frac{4\pi^2}{n^2}.$$

The Cheeger constant of C_n is $4/(n-1)$ if n is odd and $4/n$ if n is even, where the corresponding cut is given by a cycle segment of length $\lfloor n/2 \rfloor$. Thus, in any case, we have

$$d - \lambda_2 = \mathcal{O}(h(g)^2).$$

For the other bound, we consider the n-dimensional hypercube $\mathbb{H}_n$. Its eigenvalues are also known from a previous exericse, and the eigenvalue gap is

$$d - \lambda_2 = n - (n-2) = 2.$$

However, by considering $S = \{x \in \{0,1\}^n : x_0 = 0\}$, we see that

$$h(\mathbb{H}_n) \le e(S, \overline{S})/|S| = 1,$$

so in this example, we have $d - \lambda \ge 2h(\mathbb{H}_n)$.

Solution to Exercise 2

If $d = 1$, then the graphs must all be perfect matchings, which are not even connected (so $d - \lambda = 0$).

If $d = 2$, then the graphs must be a disjoint unions of cycles. For a graph to be an expander, it must additionally be connected, so in fact each graph in the family must be a cycle. However, cycles are not expanders, as we proved in Exercise 1 that $d - \lambda \le 4\pi^2/n^2 \to 0$.

Solution to Exercise 3

Let $\mu \in \mathbb{R}^n$ be the vector in which all entries are 1. Let A_G be the adjacency matrix of G. We know that $A_G\mu = d\mu$, and that d is the largest eigenvalue of A_G. Let $v^{(1)}, \ldots, v^{(n)}$ be a basis of orthogonal eigenvectors of A_G with corresponding eigenvalues $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$, where $v^{(1)} = \mu$ and $\lambda_1 = d$.

Note that the adjacency matrix of $\overline{G}$ is

$$A_{\overline{G}} = \mu\mu^T - I - A_G.$$

For $i \ge 2$, we have $\mu^T v^{(i)} = 0$ and so

$$A_{\overline{G}} v^{(i)} = \mu\mu^T v^{(i)} - v^{(i)} - A_G v^{(i)} = -v^{(i)} - \lambda v^{(i)} = (-1-\lambda)v^{(i)}.$$

Moreover, as $\overline{G}$ is $(n-1-d)$-regular, we know that μ is an eigenvector of $\overline{G}$ with eigenvalue $n-1-d$.

Thus the eigenvectors of A_G are exactly the eigenvectors of $A_{\overline{G}}$, and the eigenvalues of $A_{\overline{G}}$ are

$$n-1-d \ge -1-\lambda_n \ge -1-\lambda_{n-1} \ge \cdots \ge -1-\lambda_2.$$

The first inequality holds just because $\overline{G}$ is $(n-1-d)$-regular.