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Solution 9

Randomized Algorithms and Probabilistic Methods: Advanced Topics

Solution to Exercise 1

We divide the process into two phases: the first phase when there are still isolated vertices, and the second phase when there are no such vertices any more.

First phase. Our first goal is to estimate the duration of the first phase. Let $Z_0(m)$ denote the number of isolated vertices in G_m. We have $Z_0(0) = n$ and for $x > 0$ we have

$$\mathbb{E}[Z_0(m+1) - Z_0(m) \mid Z_0(m) = x] = -1 - x/n,$$

since we will connect an isolated vertex with a randomly chosen other vertex, which is isolated with probability x/n. Thus, for $z_0(t)$ being the solution of $z_0(0) = 1$ and

$$z_0'(t) = -1 - z_0(t), \tag{1}$$

the theorem from the lecture tells us that whp

$$Z_0(tn) = nz_0(t) + o(n)$$

holds for all t such that $Z_0(t) > 0$.

We can solve the differential equation (1) using the method of variation of constants. The solution to the homogeneous system $x'(t) = -x(t)$ is $x(t) = e^{-t}$. Assuming now $z_0(t) = c(t)e^{-t}$, we obtain $z_0'(t) = c'(t)e^{-t} - z_0(t)$. To match (1) we need $c'(t)e^{-t} = -1$ and $c(0)e^{-0} = 1$, i.e., $c(t) = 2 - e^t$. Therefore $z_0(t) = (2 - e^t)e^{-t} = 2e^{-t} - 1$ is the unique solution of (1).

In particular, as $2e^{-t} - 1 = 0$ only in $t = \ln 2$, we obtain that the time when all isolated vertices are gone is close to $n \ln 2$.

To study the next phase, we also need to know the number of degree-one vertices at the end of the first phase. This can also be obtained with differential equations. First, let $Z_1(m)$ denote the number of vertices of degree one in G_m. We have $Z_1(0) = 0$ and for $x_0 > 0$,

$$\mathbb{E}[Z_1(m+1) - Z_1(m) \mid Z_1(m) = x_1 \text{ and } Z_0(m) = x_0] = 1 + x_0/n - x_1/n,$$

since every round a vertex goes from being isolated to having degree one, and there is a probability of x_0/n to turn an additional degree zero vertex to degree one, and a probability of x_1/n to turn a vertex with degree one into a vertex with degree two. We thus get the differential equation $z_1(0) = 0$ and $z_1'(t) = 1 + z_0(t) - z_1(t) = 2e^{-t} - z_1(t)$. Again the solution is obtained by variation of constants. The homogeneous system has the solution e^{-t}, so we use the Ansatz $z_1(t) = c(t)e^{-t}$, giving $z_1'(t) = c'(t)e^{-t} - z_0(t)$. This time we want $c'(t)e^{-t} = 2e^{-t}$ and $c(0)e^{-0} = 0$, i.e., $c(t) = 2t$. Thus we have

$$Z_1(tn) = nz_1(t) + o(n) = n2te^{-t}, \tag{2}$$

which is valid for all t such that $Z_0(tn) > 0$.

In particular, since the first phase whp ends around time $n \ln 2$, we get that at the beginning of the second phase, there are $n \ln 2 + o(n)$ many vertices of degree one whp.

Second phase. In the second phase, we start with no isolated vertices and $n \ln 2 + o(n)$ vertices of degree one. For $x > 0$ we have

$$\mathbb{E}[Z_1(m+1) - Z_1(m) \mid Z_1(m) = x \text{ and } Z_0(m) = 0] = -1 - x/n.$$

From this, we get the differential equation $z_1(t) = \ln 2$ and $z_1'(t) = -1 - z_1(t)$. The solution can be determined as before to be

$$z_1(t) = (1 + \ln 2)e^{-t} - 1.$$

The second phase ends whp when $z_1(t) = 0$, i.e., when

$$t = \ln(1 + \ln 2).$$