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Solution GHW 1

Randomized Algorithms and Probabilistic Methods: Advanced Topics

Solution to Exercise 1

Let R_t denote the number of red balls at time t, let B_t denote the number of blue balls at time t, and let $X_t := R_t - n$. Initially, $X_0 = s_0 := n$. Let T be the smallest integer $t > 0$ for which $X_t = 0$. We need to compute the drift

$$\Delta(a) := \mathbb{E}[X_t - X_{t+1} \mid X_t = a].$$

Consider a fixed round $t < T$. Then

$$\mathbb{E}[R_t - R_{t+1} \mid R_t] = \frac{R_t}{2n} - \frac{B_t}{2n} = \frac{X_t + n - (2n - (X_t + n))}{2n} = \frac{X_t}{n},$$

so $\Delta(a) = a/n$.

By the multiplicative drift theorem, $\mathbb{E}[T \mid X_0 = n] \leq n \ln n + n$ and

$$\Pr[T > \lceil n \ln n + cn \rceil] \leq e^{-c}.$$

Solution to Exercise 2

Let X_t be the number of pictures that Alice does not own an odd number of times. Initially, $X_0 = n$. Let T be the first round for which $X_t = 0$. The drift of X_t is

$$\Delta(a) = \mathbb{E}[X_t - X_{t+1} \mid X_t = a] = \frac{a(a-1)}{n^2} + \frac{a}{2n}$$

for all $a > 0$.

Let $c(x) = x - 2$. Then we have $X_t \geq c(X_{t+1})$. Moreover, for $a > 0$,

$$\Delta(a) \leq h(c(a)),$$

where $h(x) = \Delta(x + 2)$.

We have

$$\frac{1}{\Delta(a)} = \frac{2n^2}{2a(a-1) + na} = \frac{2n^2}{a(2a + n - 2)} = \frac{2n^2}{a(n-2)} - \frac{4n^2}{(2a + n - 2)(n-2)}$$

By the lower-bound version of the variable drift theorem,

$$\mathbb{E}[T] \geq \int_1^n \frac{\mathrm{d}x}{h(x)} = \int_3^{n+2} \frac{\mathrm{d}x}{\Delta(x)} = \frac{2n^2}{n-2}\Big[\ln(x) - \ln(x + n/2 - 1)\Big]_3^{n+2}.$$

Therefore,

$$\mathbb{E}[T] \geq \frac{2n^2}{n-2}\Big(\ln(n+2) - \ln 3 - \ln(n + 2 + n/2 - 1) + \ln(n/2 + 2)\Big) = 2n \ln n + \mathcal{O}(n).$$

The upper bound is similar.

Solution to Exercise 3

Let $g(x) = \ln x \ln\ln x$ and define

$$Y_t = \begin{cases} g(X_t) & \text{if } X_t \geq e, \\ 0 & \text{otherwise.} \end{cases}$$

Let T be the first point in time where $Y_T = 0$. Note that this is also the first point in time where X_T goes below e. Thus it suffices to show that $\mathbb{E}[T] = \Omega(\ln n \ln\ln n)$.

(a) Let us compute the drift

$$\Delta(a) = \mathbb{E}[Y_t - Y_{t+1} \mid Y_t = a].$$

Fix any $a > 0$ and let $x = g^{-1}(a)$. We have

$$\Delta(a) = a - \sum_{i=3}^{\lfloor ex \rfloor} \frac{g(i)}{1 + \lfloor ex \rfloor},$$

by conditioning on the different possible values of X_{t+1}. We can bound

$$\sum_{i=3}^{\lfloor ex \rfloor} g(i) \geq \int_e^{ex} g(z)\,\mathrm{d}z - g(ex)$$

and

$$\frac{1}{1 + \lfloor ex \rfloor} \geq \frac{1}{2 + ex} \geq \frac{1}{ex} - \frac{2}{e^2x^2}.$$

Combining everything,

$$\Delta(a) \leq a - \left(\frac{1}{ex} - \frac{2}{e^2x^2}\right)\left(\int_e^{ex} g(z)\,\mathrm{d}z - g(ex)\right) = a - \frac{1}{ex}\int_e^{ex} g(z)\,\mathrm{d}z + o(1), \tag{1}$$

as $x \to \infty$.

The next step is to bound the integral of $g(z)$. Let $a(z) = z \ln z - z$ and let $b(x) = \ln\ln z$. Observe that $a'(z) = \ln z$. Integrating by parts,

$$\int_e^{ex} g(z)\,\mathrm{d}z = \int_e^{ex} a'(z)b(z)\,\mathrm{d}z = \big[a(z)b(z)\big]_{z=e}^{ex} - \int_e^{ex} a(z)b'(z)\,\mathrm{d}z.$$

Since $a(z)b'(z) = (z\ln z - z)/(z \ln z) = 1 - 1/\ln z$, we can simply upper bound

$$\int_e^{ex} a(z)b'(z)\,\mathrm{d}x \leq \int_e^{ex} 1\,\mathrm{d}x \leq ex.$$

Therefore, we obtain

$$\int_e^{ex} g(z)\,\mathrm{d}z \geq (ex\ln(ex) - ex)\ln\ln(ex) - ex = ex \cdot g(ex) - ex\ln\ln x - ex.$$

Plugging this into (1), we get

$$\Delta(a) \leq a - g(ex) + \ln\ln x + 1 + o(1).$$

Since

$$g(ex) = (1 + \ln x)\ln(1 + \ln x) = \ln\ln x + \ln x \ln\ln x + o(1),$$

and $a = g(x) = \ln x \ln\ln x$, this gives

$$\Delta(a) \leq 1 + o(1).$$

(b) Let $C > 0$ be such that $\Delta(a) < C$ for all $a > 0$. The Theorem 1.1 immediately gives

$$\mathbb{E}[T] \geq Y_0/C = g(n)/C = (\ln n \ln\ln n)/C,$$

under the assumption that $\lim_{t\to\infty} \mathbb{E}[X_t] = 0$.

(c) Let $C > 0$ be such that $\Delta(a) < C$ for all $a > 0$. Then, as in the proof of Theorem 1.1, we have

$$\Pr[T > t] \geq \mathbb{E}[Y_t - Y_{t+1}]/C.$$

Therefore

$$C\mathbb{E}[T] \geq \sum_{t=0}^{\infty} C \Pr[T > t] \geq \sum_{t=0}^{t_0} C \Pr[T > t] \geq \sum_{t=0}^{t_0} \mathbb{E}[Y_t] - \mathbb{E}[Y_{t+1}] = \mathbb{E}[Y_0] - \mathbb{E}[Y_{t_0+1}].$$

In particular (taking lim sup on both sides),

$$C\mathbb{E}[T] \geq \mathbb{E}[Y_0] - \liminf_{t\to\infty} \mathbb{E}[Y_t].$$

Since $Y_0 = \ln n \ln \ln n$, the claim follows.

(d) Assume that $\liminf_{t\to\infty} \mathbb{E}[Y_t] > 0$. Then there must exist some δ and some positive integer t_0 such that for all $t \geq t_0$, we have $\mathbb{E}[Y_t] \geq \delta$.

We have

$$\Pr[Y_t > 0] = \frac{\mathbb{E}[Y_t]}{\mathbb{E}[Y_t \mid Y_t > 0]}.$$

Note that, deterministically,

$$Y_t \leq g(ne^t) = \ln(ne^t) \ln \ln(ne^t) = (t + \ln n) \cdot (\ln(t + \ln n).$$

Therefore, if t_0 is sufficiently large, then for all $t \geq t_0$,

$$\Pr[Y_t > 0] \geq \frac{\delta}{(2t)\ln(2t)}.$$

From this, we obtain

$$\mathbb{E}[T] = \sum_{t=0}^{\infty} \Pr[T > t] = \sum_{t=0}^{\infty} \Pr[Y_t > 0] \geq \sum_{t=t_0}^{\infty} \frac{\delta}{(2t\ln(2t)} = \infty.$$