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Solution GHW 3

Randomized Algorithms and Probabilistic Methods: Advanced Topics

Solution to Exercise 1

Let $G = (V, E)$ be a d-regular expander graph of order n with spectral expansion λ. Assume that $f\colon G \to \mathbb{R}$ is Lipschitz continuous, i.e., that there exists a constant $M \geq 0$ such that for all vertices $u, v \in V$, we have

$$|f(u) - f(v)| \leq M \cdot d(u, v),$$

where $d(u, v)$ denotes the distance between the vertices u and v on G. Let K be a median for the values $f(v)$, i.e., let K be such that

$$|v \in V : f(v) \geq K| \geq n/2 \quad \text{and} \quad |v \in V : f(v) \leq K| \geq n/2.$$

Then we claim that there exist constants $c, C > 0$ such that for all $m \geq 0$, we have

$$|v \in V : |f(v) - K| \geq m| \leq Cne^{-cm}. \tag{1}$$

Before proving this, note that this general statement immediately implies the statement in the exercise. Indeed, the eccentricity function is Lipschitz with Lipschitz constant 1, and since all eccentricities are positive integers, the median K can be chosen to be a positive integer as well.

When proving the claim, we can assume that $M > 0$, as otherwise the claim is trivially true (say, with $C = 1$). Additionally, we will for now only consider the case $m > 0$. Thus, fix some $m > 0$ and let A_0 be the set of all vertices v such that $f(v) \leq K - m$. Moreover, for every $i \in \mathbb{N}$, let A_i be the set of all vertices that have distance at most i to A_0.

By the Lipschitz condition, we immediately see that for all $v \in A_i$, we have $f(v) \leq K - m + Mi$. Let k be the largest integer strictly smaller than m/M; since we assume that $m, M > 0$, we have $k \geq 0$. Since at least $n/2$ vertices satisfy $f(v) \geq K$, we have $|A_i| \leq |A_k| \leq n/2$ for all $0 \leq i \leq k$. Thus, by Cheeger's inequality, for all $0 \leq i \leq k$, we have

$$|A_{i+1}| \geq |A_i| + d^{-1}|E(A_i, V \setminus A_i)| \geq |A_i| + (d - \lambda) \cdot |A_i|/(2d) = \left(1 + \frac{d - \lambda}{2d}\right)|A_i|.$$

By induction, we obtain

$$|A_k| \geq \left(1 + \frac{d - \lambda}{2d}\right)^k |A_0|. \tag{2}$$

However, since $|A_k| \leq n/2$, we must have

$$|A_0| \leq \frac{n}{2}\left(1 + \frac{d - \lambda}{2d}\right)^{-k}. \tag{3}$$

Since $k \geq m/M - 1$, we have in particular

$$|v \in V : f(v) \leq K - m| \leq \frac{n}{2}\left(1 + \frac{d - \lambda}{2d}\right)^{1 - \frac{m}{M}}. \tag{4}$$

The argument above also applies to the function $g(v) = -f(v)$, which is Lipschitz with the same Lipschitz constant M, and which has $-K$ as a median value. Therefore

$$|v \in V : f(v) \geq K + m| = |v \in V : g(v) \leq -K - m| \leq \frac{n}{2}\left(1 + \frac{d - \lambda}{2d}\right)^{1 - \frac{m}{M}}.$$

All in all, we have proved that for all $m > 0$, we have

$$|v \in V : |f(v) - K| \geq m| \leq n \left(1 + \frac{d-\lambda}{2d}\right)^{1-\frac{m}{M}},$$

and this is clearly also true for $m = 0$.

Solution to Exercise 2

(1) First, a note about the definition of $p_\ell(T_d)$: stricly speaking, there is no uniform distribution on $V(T_d)$ (as this is a countably infinite set), which makes the definition given in the exercise appear meaningless. However, it is clear that in T_d, the probability that a random walk of length 2ℓ is closed does not depend on the choice of the starting vertex, so the definition still makes sense.

We start by showing that $p_\ell(G) \geq p_\ell(T_d)$. This is easiest to see by a coupling argument. Observe that, by considering repeated vertices as different vertices except when moving backwards, every walk on G gives rise to a walk of the same length on T_d. Moreover, a uniformly random walk on G gives rise to a uniformly random walk on T_d in this way. If the random walk on T_d is closed, then the random walk on G is also closed, but not necessarily vice-versa. This implies that the probability that a random walk of length 2ℓ in G is closed is at least the probability that a random walk of length 2ℓ on T_d is closed.

Next, we will prove that $p_\ell(T_d) \geq C_\ell(d-1)^\ell/d^{2\ell}$. Fix any starting vertex x_1 in T_d. Clearly, there are $d^{2\ell}$ different walks of length 2ℓ that start in x_1. It is also easy to see that the number of *closed* walks of length 2ℓ starting in x_1 is at least $C_\ell(d-1)^\ell$. To see this, we can interpret an opening parenthesis as moving away from x_1 and a closing paranthesis as moving towards the starting point. Since there are at least $d-1$ ways to move away from x_1 (there are exactly $d-1$ except if we are at the root, where there are d ways), we see that each properly parenthesized string in $\{(,)\}^{2\ell}$ corresponds to at least $(d-1)^\ell$ distinct closed random walks starting in x_1, and thus the total number of such walks is indeed at least $C_\ell(d-1)^\ell$. This shows that $p_\ell(T_d) \geq C_\ell(d-1)^\ell/d^{2\ell}$.

(2) Let P be the transition matrix of the random walk on G. Then $p_\ell(G)$ is the average of the diagonal elements of $P^{2\ell}$, i.e., $p_\ell(G) = \mathrm{tr}(P^{2\ell})/n$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of P. Because the trace of a matrix is the sum of its eigenvalues, we have

$$n \cdot p_\ell(G) = \mathrm{tr}(P^{2\ell}) = \sum_i \lambda_i^{2\ell} = 1 + \sum_{i \geq 2} \lambda_i^{2\ell} \leq 1 + (n-1) \cdot (\lambda/d)^{2\ell}.$$

Note that we have to divide by d because λ is computed from the eigenvalues of the adjacency matrix of G, which differ from the eigenvalues of P by a factor of d.

(3) Combining (1) and (2), we have

$$n \cdot C_\ell(d-1)^\ell/d^{2\ell} \leq 1 + (n-1) \cdot (\lambda/d)^{2\ell},$$

which gives

$$\lambda \geq \left(\frac{n \cdot C_\ell(d-1)^\ell - d^{2\ell}}{n-1}\right)^{1/2\ell}.$$

Suppose first that ℓ is a positive constant. Then as $n \to \infty$, we get

$$\lambda \geq \left(C_\ell(d-1)^\ell - o(1)\right)^{1/2\ell} = C_\ell^{1/2\ell}(d-1)^{1/2} - o(1).$$

Next, we show that $\lim_{\ell\to\infty} C_\ell^{1/2\ell} = 2$; note that by letting ℓ tend to infinity sufficiently slowly, this will give the desired lower bound on λ. We are given the expression

$$C_\ell = \binom{2\ell}{\ell} \cdot (\ell+1)^{-1}.$$

Using Stirling's approximation, as $\ell \to \infty$, we have

$$\binom{2\ell}{\ell} = \frac{(2\ell)!}{\ell!^2} \sim 2^{2\ell}/\sqrt{\pi\ell},$$

so

$$\lim_{\ell\to\infty} C_\ell^{1/2\ell} = 2 \cdot \lim_{\ell\to\infty} \left(\frac{1}{\sqrt{\pi\ell}(\ell+1)}\right)^{1/2\ell} = 2,$$

using that $\ell^{-3/4\ell} \to 1$.

Solution to Exercise 3

Recall that $v \in \mathbb{R}^n$ is defined by

$$v_i = \begin{cases} |\overline{S}| & \text{if } i \in S \\ -|S| & \text{otherwise.} \end{cases}$$

Then

$$v^T Av = \sum_{ij} a_{ij} v_i v_j = -2|E(S,\overline{S})| \cdot |S||\overline{S}| + 2|E(\overline{S})| \cdot |S|^2 + 2|E(S)| \cdot |\overline{S}|^2.$$

However, since G is d-regular, we have

$$2|E(S)| = d|S| - |E(S,\overline{S})| \quad \text{and} \quad 2|E(\overline{S})| = d|\overline{S}| - |E(S,\overline{S})|,$$

so by plugging this into the equation above, we get

$$v^T Av = -2|E(S,\overline{S})| \cdot |S||\overline{S}| + d|S|^2|\overline{S}| - |S|^2|E(S,\overline{S})| + d|\overline{S}|^2|S| - |\overline{S}|^2|E(S,\overline{S})|.$$

Therefore

$$\frac{v^T Av}{n \cdot |S||\overline{S}|} = \frac{d|S| + d|\overline{S}|}{n} - |E(S,\overline{S})| \cdot \frac{2|S||\overline{S}| + |S|^2 + |\overline{S}|^2}{n \cdot |S||\overline{S}|}.$$

Since $|S| + |\overline{S}| = n$, this simplifies to

$$\frac{v^T Av}{n \cdot |S||\overline{S}|} = d - \frac{n|E(S,\overline{S})|}{|S||\overline{S}|} = d - \phi(S).$$

Observe that v is orthogonal to the eigenvector $(1,1,\ldots,1)$ with eigenvalue d (the inner product is $|S||\overline{S}| - |\overline{S}||S| = 0$). By the Courant-Fischer inequality, we then have

$$\lambda_2 \geq \frac{v^T Av}{v^T v} = \frac{v^T Av}{|\overline{S}|^2|S| + |S|^2|\overline{S}|} = \frac{v^T Av}{n \cdot |S||\overline{S}|} = d - \phi(S).$$

Rearranging, we have $\phi(S) \geq d - \lambda_2$, as required.