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Advanced Data Structures

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Exercise Set 2

Exercise 1:
Let $u = 2^{c\ell}$. For every key $0 \leq x < u$, and $c \geq 2$. Let $h(x) = T_1(x_1) \oplus T_2(x_2) ... \oplus T_c(x_c)$, where $x_1, \ldots, x_c$ are digits of x in 2^ℓ basis, and each T_i is totally random hash function $2^\ell \to 2^{\ell'}$, for some $\ell' \leq \ell$.
Show that family of $h(x)$ is 3-wise independent, but not 4-wise independent.

Hint:
4-wise independence: it is enough to point a single quadruple of distinct keys A, B, C, D for which $h(A), h(B), h(C), h(D)$ are correlated.

3-wise independence:
Consider any triplet of keys A, B, C. Show that there is coordinate i, such that if we fix in place all hash functions except T_i, iterating over all possible values of T_i gives us identical probability for all possible values of $(h(A), h(B), h(C))$.
Useful fact: for any fixed $0 \leq y < 2^\ell$, $x \to x \oplus y$ is bijective function.

Exercise 2:
Show that the longest chain has length $\mathcal{O}(\frac{\lg n}{\lg \lg n})$ w.h.p.

Hint:
Use Chernoff bound (where μ denotes $E[X]$):

$$\Pr(X > c\mu) < \left(\frac{e^{(c-1)}}{c^c}\right)^\mu .$$

Exercise 3:
Consider a Cuckoo hashing. Show that if f and g used in hashing are totally random functons, and $m \geq 2n$, then

$$\Pr(\text{Insert follows bump path of length } k) \leq 1/2^k).$$