Exercise Set 1 – FS17
(Linear Algebra Methods in Combinatorics)

These exercises are **non-graded** but you get feedback on your submitted solutions. You can submit solutions by next exercise class, **1.3.2017**, also by **email** (barbara.geissmann@inf.ethz.ch).

Exercise 4. *Consider the "Reversed Oddtown" problem that is the variant of Oddtown in which we swap "odd" with "even": we want* $S_1, \dots, S_m \subseteq [n]$ *such that*

$$|S_i| \text{ is even for all } i \tag{17}$$

$$|S_i \cap S_j| \text{ is odd for all } i \neq j \tag{18}$$

Show how the result of Oddtown implies $m \leq n+1$ *for this problem.*

Exercise 5. *Construct a solution for the "mod-4-town" problem in Exercise 9 such that the corresponding incidence vectors are linearly* ***dependent*** *over* $\mathbb{F}_2$.

Exercise 6. *Consider these two polynomials in one variable*

$$f_1(x) = x^2 \quad \text{and} \quad f_2(x) = 1 - 2x$$

over the field $\mathbb{R}$. *Show that they are linearly independent.*

Exercise 7. *In this exercise we consider functions that are* ***polynomials*** *over the reals of* ***degree at most*** d. *In particular, we have* $f_1, \dots, f_m : \mathbb{R} \to \mathbb{R}$ *and these polynomials satisfy the following additional condition: there are numbers* $s_1, \dots, s_m \in \mathbb{R}$ *such that*

$$f_i(s_i) \neq 0 \qquad \text{for every } i \tag{19}$$

$$f_i(s_j) = 0 \qquad \text{for every } i \neq j. \tag{20}$$

Prove that these polynomials are at most $d+1$ *using Theorem 2 in the lecture notes. (**Hint:** you may try first with* $m = 2$ *and look back at the solution of the previous exercise.)*

The following two exercises have been discussed already during the exercise class (22.2.2018), but you can submit solutions to these as well, if you want to have feedbacks.

Exercise 8. *Consider the following variant of Oddtown: we want* $S_1, \dots, S_m \subseteq [n]$ *such that*

$$|S_i| \neq 0 \ mod\ 3 \qquad \textit{for every } i \tag{21}$$
$$|S_i \cap S_j| = 0 \ mod\ 3 \qquad \textit{for every } i \neq j. \tag{22}$$

Prove that $m \leq n$. *Prove the same result for the version in which "mod 3" is replaced by "mod p" with p be any prime.*

Exercise 9. *Call mod-*p^k*-town the variant of Oddtown in which "mod 2" is replaced by "mod* p^k*" for prime* p *and positive integer* k. *Show that in this variant* $m(n) \leq n$. *(**Hint:** show linear independence over* $\mathbb{Q}$ *by adapting the solution of mod-4-town seen in the exercise class)*