Exercise Set 4 – FS18
(Linear Algebra Methods in Combinatorics)

These exercises are **non-graded** but you get feedback on your submitted solutions. You can submit solutions by next exercise class, **22.3.2017**, also by **email** (**barbara.geissmann@inf.ethz.ch**).

Exercise 1 (**1 point**). *Suppose $S_1, \dots, S_m \subseteq \{1, \dots, n\}$ satisfy the conditions of Fisher inequality:*

$$|S_i \cap S_j| = c \quad \text{for all } i \neq j$$

Show that if one subset S_k has size $|S_k| = c$ then it must be $m \leq n$.

Exercise 2 (**3 points**). *Write an alternative proof of Fisher inequality based on the following notion:*

> A matrix A is said positive **semi**definite if
>
> $$x^T A x \geq 0 \quad \text{for all } x \neq \mathbf{0}$$
>
> and it is said positive **definite** if
>
> $$x^T A x > 0 \quad \text{for all } x \neq \mathbf{0}$$

Show that the matrix $A = (a_{ij})$ with $a_{ij} = v_i \cdot v_j$ is positive definite and explain how from this you obtain Fisher inequality.
Note: The vectors v_i's are the usual incident vectors of the subsets. As in the proof in the notes you can assume that each subset has size strictly larger than c, i.e., $|S_i| > c$ for all i.

Exercise 3 (**2 points**). *Extend the idea of the cubic construction (Section 2 in the lecture notes) to obtain the following: A construction of a 3-coloring of the complete graph with $n = \binom{t}{5}$ nodes so that there is no monochromatic complete subgraph of size $t + 1$.*
Note: By "3-coloring" we mean that we color the edges using colors red, blue, and green.

Exercise 4 (2 points). *It is possible to extend the Ramsey Theorem for two colors in the lecture notes as follows:*

> **Theorem.** For every natural numbers c and t, there exists a natural number $n = R(t; c)$ such that, if we color the complete graph with n or more nodes using c **colors**, then there must be monochromatic complete subgraph of size t.

Show that the Ramsey Theorem for c colors (theorem above) implies the following:

> **Theorem.** For every c there exists a natural number $n = S(c)$ such that, if we color the integers $\{1, 2, ..., n\}$ using c colors, then there exist integers a, b and $a + b$ that get the same color.

Hint: Look at what happens if we color the edges of a complete graph so that the color of every edge (u, v) depends only on the difference $u - v$ (consider the vertices as integers). Note that the theorem does not assume $a \neq b$.