Exercise Set 8 – FS18
(Linear Algebra Methods in Combinatorics)

You can submit solutions **also by email** by next lecture – **26.4.2018**. These exercises are **non-graded** but you get feedback on your submitted solutions.

Some exercises on Lecture 7.

Exercise 1. *Recall that a family is said* ***critical*** *if the following two conditions hold (see lecture notes):*

CR1: *We need $s+1$ nodes to cover all of its members;*

CR2: *As soon as we remove* ***any one*** *member from the family, then s nodes are enough.*

Bollobás uniform theorem (Theorem 3 in Lecture 7) implies that no r-uniform critical family can have more than $\binom{s+r}{r}$ members. Prove that this result is false for r-uniform families that satisfy ***only CR1****: show that it is possible to construct an* ***arbitrarily large*** *r-uniform family satisfying* ***CR1 only*** *and such that $s+1$ nodes are enough to cover all members of the family.*

Exercise 2. *Let $\mathcal{F}$ be an r-uniform set system of size larger than $\binom{s+r}{r}$. Prove that there exists one $A \in \mathcal{F}$ such that*

$$\mathcal{F}' := \mathcal{F} \setminus A \quad \text{and} \quad \mathcal{F}$$

have the ***same covering number****, that is, they can be both covered by s nodes, and $s-1$ are not enough.*

Exercise 3. *Construct a critical set for $r=2$ and $s=2$ matching the bound of Bollobás theorem.*

Some exercises on Lecture 8.

Exercise 4. *Give a formal proof of (11) used in the Proof of Claim 3 in the lecture note. We restate (11) here for convenience:*

If p is a polynomial over $\mathbb{F}_2$ in n variables and of degree at most $2k - n - 1$, then

$$\sum_{\substack{y\in\{0,1\}^n \\ y\le x,\ y\le x'}} p(y) = 0$$

where x and x' are two inputs with exactly k 1's, for $k > n/2$, and '$y \le x$' is the bitwise 'less than or equal' (similarly for $y \le x'$).

Exercise 5. *In the proof of Claim 3 I'm tempted to take this "**shortcut**":*

For any x and x'

$$\sum_{\substack{y\in DIFF \\ y\le x}} m_{x',y} = \sum_{\substack{y\in DIFF \\ y\le x,\ y\le x'}} 1 \;=\; \sum_{\substack{y\in DIFF \\ y\le x,\ y\le x'}} f_k(y) + p(y) \tag{9}$$

*The **first term in (9) produces the identity matrix:***

$$\sum_{\substack{y\in DIFF \\ y\le x,\ y\le x'}} f_k(y) = \begin{cases} 1 & \text{for } x' = x \\ 0 & \text{for } x' \neq x \end{cases} \tag{10}$$

*while **the second term in (9) is always** 0**:***

$$\sum_{\substack{y\in DIFF \\ y\le x,\ y\le x'}} p(y) = 0 \tag{11}$$

By putting the three equations together we get Claim 3.

Explain why this proof is wrong and where it does not work.