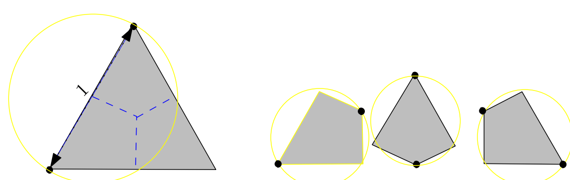
Lecture 11

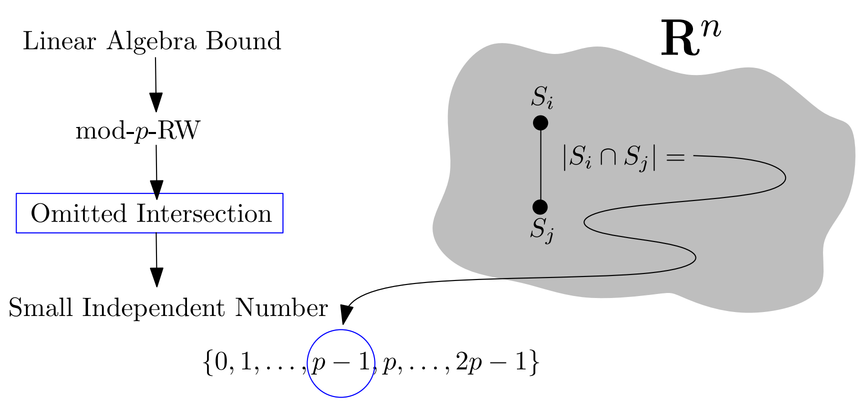
Linear Algebra Methods in Combinatorics

with Applications to Geometry and CS



Tools from the previous lectures

Recall the **unit-distance** graph G_p that we constructed in $\mathbb{R}^n$ that has a high chromatic number:



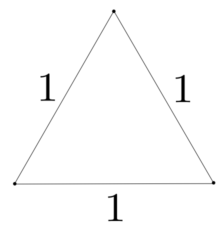
$$\boxed{\text{Vertices} \equiv \text{Subsets}}$$

1 Reduce the diameter of bodies

This lecture deals with this problem:

> **Borsuk** Problem:
>
> Can we partition any set of diameter 1 in $\mathbb{R}^d$ into $d+1$ subsets each of them having **smaller** diameter?

The $d = 1$ case is (almost) obvious. For $d = 2$ you need at least $d+1$ pieces:

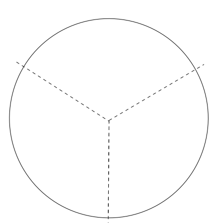


because if you break this triangle into just two pieces, than one piece contains two vertices and their distance is still 1 (the diameter).

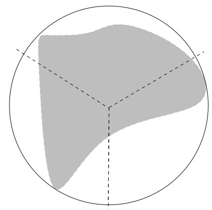
Exercise 1. *Prove that $d+1$ pieces are necessary.*

> **Borsuk Conjecture**
>
> The minimum number of pieces is $f(d) = d+1$

(Wrong) Proof for $d = 2$. Three pieces are enough for the disk:



Every body of diameter 1 in $\mathbb{R}^2$ can be inscribed into this circle:



and thus three pieces are enough in general.

Exercise 2. *Find the mistake in this proof.*

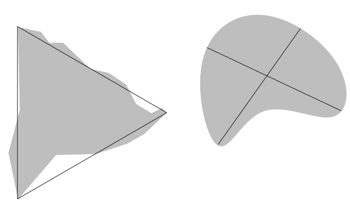
The proof is false, but the conjecture is **true** for $d = 2$ and $d = 3$ (no proof here).

2 Disprove Borsuk Conjecture

For d large enough
$$f(d) > 1.2^{\sqrt{d}}$$

2.1 Reduce to Graph Coloring

We must "separate" all points at distance 1 in our body. This means that any two such points **cannot be in the same piece** (look back at the triangle example). We can define a graph G_B consisting of all pair of points at "maximum" distance in B



Diameter Graph G_B

- Vertices $\equiv$ Points of B $\quad (B \subset \mathbb{R}^d)$
- Edges $\equiv$ "pairs at maximum distance in B"

More precisely, this is how we define the edges of our graph G_B:

$$b \text{ and } b' \text{ adjacent in } G_B \Leftrightarrow d(b,b') = diam(B) \tag{1}$$

where $diam(B) \stackrel{\triangle}{=} \sup\{d(x,y)| \;\; x,y \in B\}$.

Exercise 3. *Prove that the minimum number of pieces necessary to reduce the diameter of B is at least $\chi(G_B)$.*

$$f(d) \geq \chi(G_B) \tag{2}$$

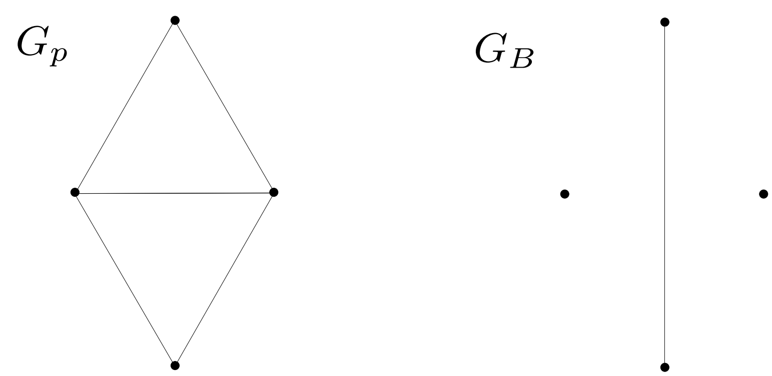
2.2 Naive idea

We have already constructed a graph G_p which has high chromatic number.

Unit-distance Graph G_p

- Vertices $\equiv$ Points (in $\mathbb{R}^n$)
- Edges $\equiv$ "pairs at distance 1" (Euclidean distance)

Unfortunately, **the two graphs are not the same**:

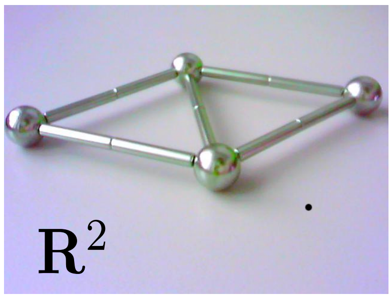


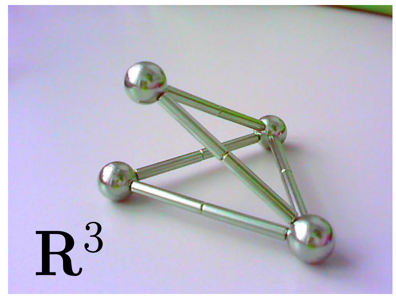
The set of points is the same, but the edges are not.

2.3 Go to higher dimensions

Look at this picture:

G_p $\qquad\qquad$ G_B





The graph on the left is a **unit distance** graph on the **plane**. The graph on the right is a **diameter graph** in the **3D space** and...**the two graphs are the same!**

Our Strategy

1. Map the points of G_p in the d-dimensional space for some $d \sim n^2$ (this gives the set B)
2. Show that $G_p \cong G_B$ (isomorphic)
3. Conclude that

$$f(d) \geq \chi(G_B) = \chi(G_p) > 1.13^n > 1.13^{\sqrt{d}}$$

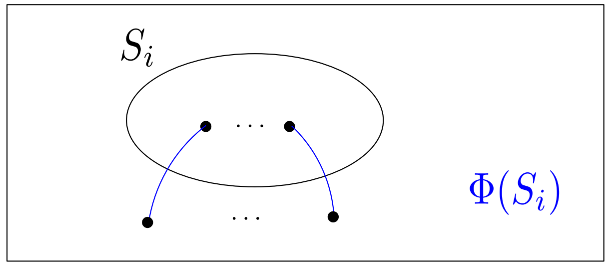
(we shall get something slightly better)

2.3.1 The mapping

Recall how we defined the vertices of G_p:

Vertices $\equiv$ Subsets

We have $n = 4p - 1$ and the vertices correspond to all subsets of size $k = 2p - 1$. Our mapping can be informally described as mapping every S_i into the subset of all "edges" with one endpoint in S_i and the other endpoint not in S_i:



$[n]$

Formally

$$\Phi(S_i) \triangleq \{\{x,y\}|\ x \in S_i \text{ and } y \in [n] \setminus S_i\}$$

Also these subsets have **uniform size** since $|\Phi(S_i)| = k(n-k)$. We can see each $\Phi(S_i)$ as a 0/1 vector of length $\binom{n}{2}$ and thus

B is the set of all points $\{\Phi(S_i)\} \in \mathbb{R}^d$ with $d = \binom{n}{2}$

2.3.2 G_p and G_B are isomorphic

Recall that

S_i and S_j adjacent in $G_p \iff |S_i \cap S_j| = p - 1$

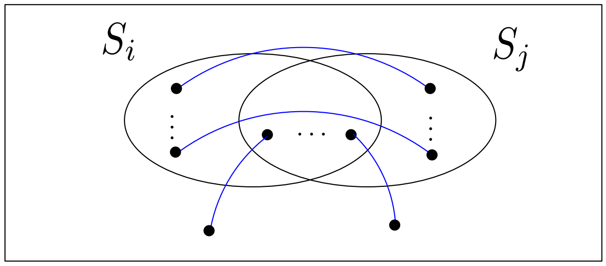
Since all $\Phi(S_i)$ have **uniform size** $h = k(n-k)$ the Euclidean distance satisfies

$$d(\Phi(S_i), \Phi(S_j))^2 = 2(h - |\Phi(S_i) \cap \Phi(S_j)|)$$

and the **diameter** of B is determined by the pairs maximizing such distance. That is, by the pairs **minimizing their intersection size**.

$\Phi(S_i)$ and $\Phi(S_j)$ adjacent in $G_B \iff |\Phi(S_i) \cap \Phi(S_j)|$ is minimum

Now we look at the size of the intersections of the new sets. The following figure shows all elements in $\Phi(S_i) \cap \Phi(S_j)$:



$[n]$

and it should convince you that for $r = |S_i \cap S_j|$ we have

$$|\Phi(S_i) \cap \Phi(S_j)| = r(n - 2k + r) + (k - r)^2$$

This can be rewritten as

$$|\Phi(S_i) \cap \Phi(S_j)| = 2(r - (k - n/4))^2 - (k^2 + n^2/8 - kn)$$

which is minimized for $r = k - n/4 = 2p - 1 - (4p - 1)/4) = p - 3/4$. Since r must be integer we get

$$\boxed{|S_i \cap S_j| = p - 1 \iff |\Phi(S_i) \cap \Phi(S_j)| \text{ is minimum}}$$

which then gives us $G_p \cong G_B$.

2.3.3 Conclusion

Since $d = \binom{n}{2}$ we have $n > \sqrt{2d}$ and thus

$$f(d) \geq \chi(G_B) = \chi(G_p) > 1.1397^n > 1.1397^{\sqrt{2d}} > 1.203^{\sqrt{d}}$$

for all d of the form $d = \binom{4p-1}{2} = (4p - 1)(2p - 1)$.

Exercise 4. *Deduce $f(d) > 1.2^{\sqrt{d}}$ for any (sufficiently large) d from the Prime Number Theorem.*

3 What to remember and where to look

The proof of "Borsuk Conjecture Disproved" can be found in [BF92, Sect. 5.6]. There you find a number of "funny" facts about this conjecture, among others: the conjecture is open already for $d = 4$. And the proof we have seen will disprove the conjecture only for $d = 1325$. The current best known result is that the conjecture is false for $d = 65$ [Bon14], and $d = 64$ [JB14].

References

[BF92] L. Babai and P. Frankl. *Linear algebra methods in combinatorics with applications to geometry and computer science*. The University of Chicago, 1992.

[Bon14] A. Bondarenko. On Borsuk's conjecture for two-distance sets. *Discrete and Computational Geometry*, 51:509–515, 2014.

[JB14] Thomas Jenrich and Andries E. Brouwer. A 64-Dimensional Counterexample to Borsuk's Conjecture. *Electr. J. Comb.*, 21(4):P4–29, 2014.

Exercises
(during next exercise class - 17.5.2018)

We shall discuss and solve together the following exercise:

Exercise 1. *I can easily disprove Borsuk's conjecture, that is, show that*

$$f(d) \geq d + 2$$

with this simple idea.

Take a ***two-distance*** *set of points,*

$$d(p_i, p_j) \in \{\delta_1, \delta_2\} \qquad \text{for all } i \neq j$$

and observe that the diameter is the largest of the two distances ($\delta_2 > \delta_1$).

The resulting diameter graph G_B will contain a clique of size K. Therefore just to separate these points, we have

$$f(d) \geq K.$$

To disprove the conjecture it is enough to show that there is a clique of size $K \geq n + 2$.

Question: *Why the above strategy* ***cannot*** *disprove the conjecture?*

Exercise Set 11 – FS18
(Linear Algebra Methods in Combinatorics)

You can submit solutions **also by email** by the next lecture – **24.5.2018**. These exercises are **non-graded** but you get feedback on your submitted solutions.

Two exercises on disproving Borsuk conjecture for a small dimension.

Exercise 1. *A* ***strongly regular*** *graph with parameters* $(n, \delta, \lambda, \mu)$ *is an undirected graph with* n *nodes, each node has degree* δ *and such that*

1. *Any two neighbors have* λ *common neighbors;*
2. *Any two non-neighbors have* μ *common neighbors.*

Show that, for any strongly regular graph with parameters as above, you can construct a ***spherical two-distance*** *set in* $\mathbb{R}^n$*, that is, a subset of points in the sphere* $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : ||x|| = 1\}$ *that for a two-distance set.*

Exercise 2. *There exists a graph with parameters* $(416, 100, 36, 20)$ *which does not contain any clique of size* 6*. Consider the diameter graph* G_B *derived from the two-distance set in the previous exercise. Prove the following:*

- *There exists a spherical two-distance set in* $\mathbb{R}^{400}$ *which requires at least* $n/5 = 416/5$ *pieces to reduce its diameter.*