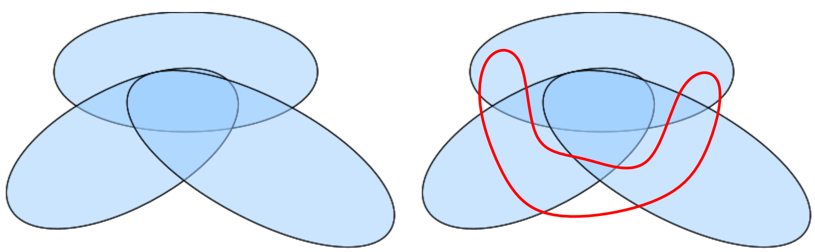
# Lecture 6
# Linear Algebra Methods in Combinatorics

**with Applications to Geometry and CS**



## 1 Helly's Theorem

The three **convex** objects in the figure above (left) intersect all in a common point (the intersection of the three of them is non-empty). Can you have **four** convex objects such that **every triple of them intersects** in a common point but **no point belongs to all of them**? The red object in the right picture is not convex.

Convexity can be defined in terms of "special" linear combinations:

We say that $\lambda_1 v_1 + \cdots + \lambda_m v_m$ is a **convex combination** if

$$\lambda_1 + \cdots + \lambda_m = 1 \quad \text{and} \quad \lambda_i \geq 0$$

(Here $v_i \in \mathbb{R}^n$ are vectors and $\lambda_i \in \mathbb{R}$.)

We denote the set of **all convex combinations** of vectors $v_1, \dots, v_m \in \mathbb{R}^n$ as

$$\mathrm{conv}(v_1, \dots, v_m)$$

which is the "analogous" of the span, though restricted to convex combinations. Point and vectors are the same, so we often talk about a set of points $S = \{v_1, \dots, v_m\}$, and write "conv($S$)" in place of "conv($v_1, \dots, v_m$)"; this is the so called **convex hull** of the set of points $S$.

A **convex object** (or convex set) is a set $C \subseteq \mathbb{R}^n$ which is closed under linear combinations (every convex combination of $v_1, \dots, v_k \in C$ belongs also to the set $C$).

**Theorem 1** (Helly's Thm. – dummy version)**.** *If $C_1, C_2, C_3, C_4$ are convex objects in 2D and any three of them have non-empty intersection, then the intersection of all of them is also non-empty.*

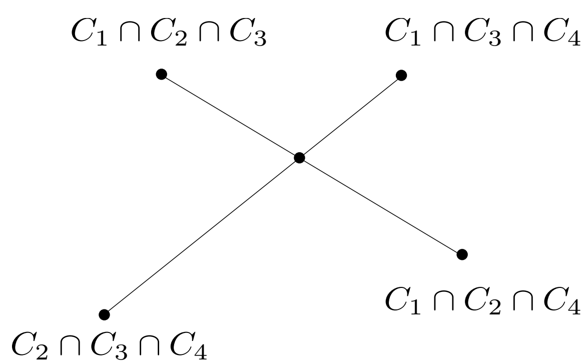
*Proof Idea – has a mistake.* We have four intersections (one for each "omitted" object) and each of them must contain a point (the intersections of three objects are non-empty):

$$\begin{aligned} I_1 &:= C_2 \cap C_3 \cap C_4 \quad \rightarrow \quad p_1 \\ I_2 &:= C_1 \cap C_3 \cap C_4 \quad \rightarrow \quad p_2 \\ I_3 &:= C_1 \cap C_2 \cap C_4 \quad \rightarrow \quad p_3 \\ I_4 &:= C_1 \cap C_2 \cap C_3 \quad \rightarrow \quad p_4 \end{aligned}$$

Each segment between two points belongs to the corresponding intersections:

$$segment(p_1, p_2) \in I_1 \cap I_2$$

Finally, we can draw two such segments that must also intersect in some point $p$. This common point will belong to **all** objects:



$\square$

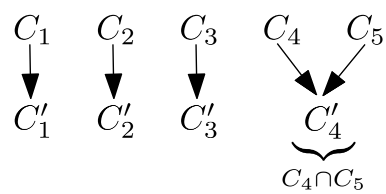
**Exercise 1.** *Find the mistake in the proof above.*

Surprisingly the same is true for **any number** of objects:

**Theorem 2** (Helly's Thm. in 2D)**.** *If $C_1, C_2, \ldots, C_m$ are convex objects in 2D and any three of them intersect (in a common point), then all of them intersect (the common intersection is non-empty).*

*Proof Idea.* The proof is by induction on $m$ and it uses the fact that the intersection of two convex objects is also convex (see Exercise 2).

To get the idea we prove the case $m = 5$. We reduce to the previous theorem ($m = 4$) by replacing the last two objects (or any two of them) by their intersection:




In order to apply the theorem for $m' = 4$ we need two things:

- $C_4'$ is also convex (Exercise 2);
- Any three sets among $C_1', C_2', C_3', C_4'$ intersect (**Exercise:** use Theorem 1 to prove that, for instance, $C_1', C_2'$ and $C_4'$ intersect).

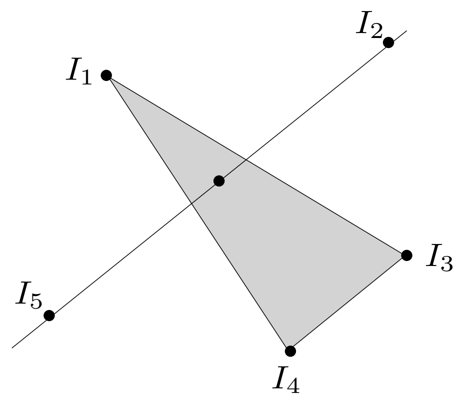
Theorem 1 says that there is a point

$$p' \in C_1' \cap C_2' \cap C_3' \cap C_4' = C_1 \cap C_2 \cap C_3 \cap C_4 \cap C_5$$

and thus the intersection of our original 5 points is nonempty. $\square$

## 1.1 Towards higher dimensions...

What happens in 3D? We have 5 objects and thus 5 points (coming from the intersections of all but one subset):




We can find 3 points whose convex hull intersects the line through the remaining two points.

**Lemma 3** (Radon)**.** *Let $S$ is a set of $m \geq d+2$ points in $\mathbb{R}^d$. Then $S$ has two disjoint subsets $S_A$ and $S_B$ whose convex hulls intersect.*

*Proof.* Look at the matrix (our points are vectors in $\mathbb{R}^d$)

$$M = \left(\begin{array}{ccc} | & & | \\ p_1 & \cdots & p_{d+2} \\ | & & | \\ \hline 1 & \cdots & 1 \end{array}\right)$$

and simply because this is an $(d+1) \times (d+2)$-matrix, there is a vector $\lambda \neq \mathbf{0}$ such that $M\lambda = 0$. That is,

$$\lambda_1 p_1 + \cdots + \lambda_{d+2} p_{d+2} = \mathbf{0} \quad \text{with} \sum_i \lambda_i = 0$$

and at least one $\lambda_i \neq 0$. The partition is

$$S_A = \{p_a | \lambda_a > 0\} \qquad S_B = \{p_b | \lambda_b < 0\}$$

(Can you tell why these subsets are nonempty?)
So we have

$$\sum_{a \in A} \lambda_a p_a + \sum_{b \in B} \lambda_b p_b = \mathbf{0}$$

and

$$\sum_{a \in A} \lambda_a + \sum_{b \in B} \lambda_b = 0$$

Take $\lambda_b^+ \triangleq -\lambda_b$ for those in $S_B$ and obtain

$$\sum_{a\in A} \lambda_a p_a = \sum_{b\in B} \lambda_b^+ p_b$$

and

$$\underbrace{\sum_{a\in A} \lambda_a}_{sum_A} = \underbrace{\sum_{b\in B} \lambda_b^+}_{sum_B}$$

Since $sum_A = sum_B$ we can obtain **convex** combinations by dividing both sides by this quantity:

$$\sum_{a\in A} \frac{\lambda_a}{sum_A} p_a = \sum_{b\in B} \frac{\lambda_b^+}{sum_B} p_b \tag{1}$$

(Can you see that these are convex combinations?)
This tells us that there is a point $p \in \text{conv}(S_A) \cap \text{conv}(S_B)$ and thus these convex hulls intersect. $\square$

**Theorem 4** (Helly)**.** *If $C_1, C_2, \ldots, C_m$ are convex objects in $\mathbb{R}^d$ with $m \geq d+2$ and any $d+1$ of them intersect (in a common point), then all of them intersect (the common intersection is non-empty).*

*Proof Sketch.* We first prove the case $m = d+2$ ("dummy version"). We have the following steps:

1. Map each object $C_i$ into an intersection which "excludes $C_i$":

$$\begin{array}{rlcl} C_1 \to & I_1 := C_2 \cap C_3 \cap \cdots \cap C_{d+2} & \to & p_1 \\ C_2 \to & I_2 := C_1 \cap C_3 \cap \cdots \cap C_{d+2} & \to & p_2 \\ \vdots & & & \\ C_{d+2} \to & I_{d+2} := C_1 \cap C_2 \cap \cdots \cap C_{d+1} & \to & p_{d+2} \end{array}$$

2. Apply Radon's Lemma to this set of points:

$$\text{conv}(S_A) \cap \text{conv}(S_B) \neq \emptyset$$

which means that there is a point $p$ with

$$p \in \text{conv}(S_A) \quad \text{and} \quad p \in \text{conv}(S_B) \tag{2}$$

3. We claim that for any $p_a \in S_A$ and $p_b \in S_B$

$$\mathrm{conv}(S_A) \subseteq C_b \quad \text{and} \quad \mathrm{conv}(S_B) \subseteq C_a \tag{3}$$

(Exercise)

4. By the previous two items we have

$$p \in \mathrm{conv}(S_A) \subseteq C_b \quad \text{and} \quad p \in \mathrm{conv}(S_B) \subseteq C_a \ . \tag{4}$$

for every $p_a \in S_A$ and $p_b \in S_B$. Since $S_A$ and $S_B$ are a partition of $S$, we have that $p \in C_i$ for every $i$, that is

$$p \in C_1 \cap \cdots \cap C_m$$

Finally the case $m > d + 2$ can be proved by induction on $m$ (Exercise). □

## 2 "Helly-theorems" for graphs

Here is a simple example of a "combinatorial version" of Helly's Theorem:

**Example 5.** *Every **three** edges of a graph intersect in one node* $\Longrightarrow$ ***All edges** intersect in one node.*

Note that **each triple can be covered by a different node**, so it is not obvious that **one node can cover all of them**: if we have triples of edges

$$T_1, T_2, \ldots$$

we only know that some node $n_1$ covers $T_1$ and (a possibly different) node $n_2$ covers $T_2$, and so on...

We can see this as a **vertex cover** problem: (a) For every three edges one vertex is enough to cover them, implies (b) The graph has a vertex cover of size one. The following theorem generalizes the previous example (no proof for now):

**Theorem 6** (Erdös-Hajnal-Moon)**.** *If every subset of $\binom{s+2}{2}$ edges of a graph can be covered by $s$ nodes, then all edges of the graph can be covered by $s$ nodes.*

One can go even further and replace edges (2-subsets) by **hyperedges** of uniform size ($r$-subsets). The hypergraph is just a collection of subsets of exactly $r$ nodes, also called $r$**-uniform set system**:[1]

**Theorem 7** (Bollobás)**.** *If every subfamily of at most $\binom{s+r}{r}$ members of an $r$-uniform set system can be covered by $s$ nodes, then all members can.*

We shall prove Bollobás Theorem in the next lecture. For the moment observe that these "Helly-type" theorems are of the form

| Local Condition $\Longrightarrow$ Global Condition |
|---|

# 3 Exercises

**Exercise 2.** *Prove that the intersection of two convex objects $C_1$ and $C_2$ is also convex.*

**Exercise 3.** *Prove Theorem 2. Proceed by induction on $m$ adapting the proof given for $m = 5$:*

1. *Reduce the number of objects to $m' = m - 1$;*
2. *Explain why you can apply the inductive hypothesis to these $m'$ objects;*
3. *Derive from the inductive hypothesis that the original $m$ objects intersect in a common point.*

*Also explain the base case of the induction.*

**Exercise 4.** *Prove Helly's Theorem 4 using Radon's Lemma 3 for the case $d = 3$ and $m = 5$ objects. That is, suppose $C_1, \ldots, C_5$ are convex objects in $\mathbb{R}^3$ and every 4 of them intersect. Show that they all intersect.*

**Exercise 5.** *Prove the result mentioned in Example 5.*

More exercises will be in the "Exercise Set 6".

[1]An $r$-uniform set system is a family $\mathcal{F} = \{S_1, \ldots, S_m\}$ where $S_i \subseteq [n]$ and $|S_i| = r$ for all $i$. This is just a "set of subsets" and we call it **family** just to avoid confusion; any subset of $\mathcal{F}$ gives another family consisting of some of the members of $\mathcal{F}$.

# Exercises
# (during next exercise class - 29.3.2018)

We shall discuss and solve together the following exercise:

**Exercise 6.** *Consider the following generalization of the statement in Example 5:*

> *Every subset of $s+2$ edges (in a given graph) can be covered by $s$ nodes $\Longrightarrow$ **All edges** can be covered by $s$ nodes (the graph has a vertex cover of size $s$).*

*This statement is false for all $s \geq 2$. Disprove it.*