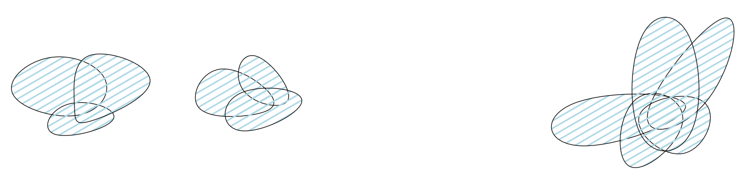
Lecture 7

Linear Algebra Methods in Combinatorics

with Applications to Geometry and CS

From the previous lectures

Local Condition $\Longrightarrow$ Global Condition
(Helly's Theorem)



And this might be also useful to get some intuition (without proof).

For every subspace U of a vector space V

$$dim(U) + dim(U^{\perp}) = dim(V)$$

1 Bollobás Theorem

This is a "Helly-type" theorem for graphs:

Theorem 1 (Erdös-Hajnal-Moon). *If every subset of $\binom{s+2}{2}$ edges of a graph can be covered by s nodes, then all edges of the graph can be covered by s nodes.*

We generalize vertex cover on graphs by replacing edges (2-subsets) with **hyperedges** of uniform size (r-subsets):

A r-**uniform set system** is a family $\mathcal{F} = \{R_1, \ldots, R_m\}$ where

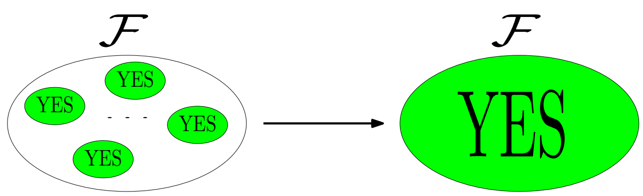
$$R_i \subseteq [n] \text{ and } |R_i| = r \text{ for all } i$$

The family $\mathcal{F}$ can be **covered** by s nodes if there is a subset $S \subseteq [n]$ of size s that intersects all members of $\mathcal{F}$

$$R_i \cap S \neq \emptyset, \qquad \text{for all } R_i \in \mathcal{F}.$$

Theorem 2 (Bollobás). *If every family of at most $\binom{s+r}{r}$ members of an r-uniform set system can be covered by s nodes, then all members can.*

Now we show the "strategy" to prove the theorem. Our theorem can be "graphically" represented as

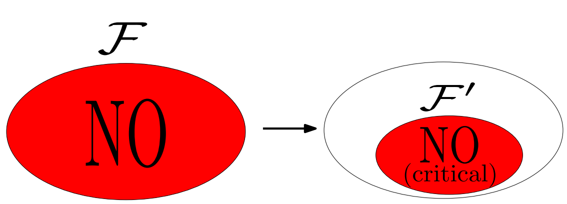


where the promise is that "all small pieces" can be covered by s nodes. Suppose we cannot cover our family using s nodes (so we need $s + 1$ or more). We start removing, one by one, members from the family until "s nodes are almost enough":

A family is said **critical** if

CR1: We need $s + 1$ nodes to cover all of its members;

CR2: As soon as we remove **any one** member from the family, then s nodes are enough.



The idea to prove Bollobás theorem is **every** critical family must be "small":

Bollobás theorem - restated

Theorem 3. *If* $R_1, \ldots, R_m \subseteq [n]$ *are subsets of size* r *and* $S_1, \ldots, S_m \subseteq [n]$ *are subsets of size* s *such that*

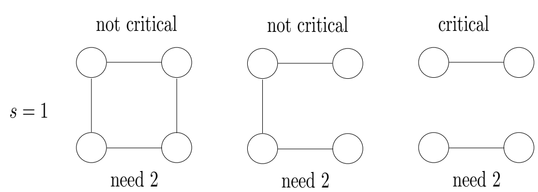
$$R_i \cap S_i = \emptyset \quad \text{for all } i \tag{1}$$

$$R_i \cap S_j \neq \emptyset \quad \text{for all } i \neq j \tag{2}$$

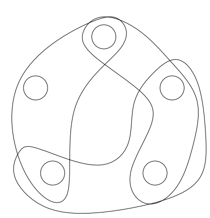
then $m \leq \binom{r+s}{r}$.

Exercise 1. *Show that Theorem 3 implies Theorem 2.*
<u>*Hint:*</u> *If we remove* R_1 *from the family, then some* S_1 *of size* s *will cover the remaining* $S_2, \ldots, S_m$.

Here is an example of (non-)critical graphs

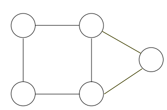


and one of a critical hypergraph ($r = 3$)



Exercise 2. *Find the largest critical graph for* $s = 1$.

Exercise 3. *Show that this graph is not critical:*



2 Proof of Bollobás Theorem

A "magic" matrix. We shall use the following

$$M = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 4 & \dots & 2^r \\ \vdots & & & & \\ 1 & i & i^2 & \dots & i^r \\ \vdots & & & & \\ 1 & n & n^2 & \dots & n^r \end{pmatrix} \tag{3}$$

The i^{th} row of this matrix is

$$m_i \triangleq (1, i, i^2, \dots, i^r)$$

which is a vector in $\mathbb{R}^{r+1}$. The main feature of this matrix is

Claim 4. *Any subset of $r+1$ rows of M are linearly independent.*

Proof. Appendix A. □

Proof of Theorem 3. In the sequel we let A and B be subsets of size r and s respectively:

$$A = \{a^1, \dots, a^r\} \quad \text{and} \quad B = \{b^1, \dots, b^s\}$$

This gives two sets of vectors (the rows of matrix M indexed by these sets):

$$V_A \triangleq \{m_{a^1}, \dots, m_{a^r}\} \quad \text{and} \quad V_B \triangleq \{m_{b^1}, \dots, m_{b^s}\}$$

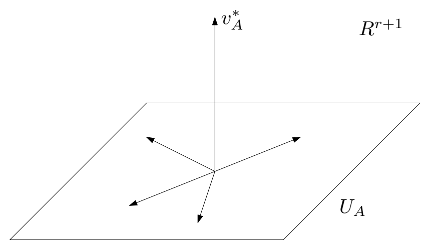
Since A has size r the subspace

$$U_A \triangleq \text{span}(m_{a^1}, \dots, m_{a^r})$$

has dimension

$$dim(U_A) = r$$

and therefore there is some "special" vector v_A^* which is **orthogonal** to all vectors in U_A (**Exercise!**).



Claim 5.

$$A \cap B = \emptyset \implies v_A^* \not\perp m_{b^k} \quad \textit{for all } b^k \in B \tag{4}$$
$$A \cap B \neq \emptyset \implies v_A^* \perp m_{b^k} \quad \textit{for some } b^k \in B \tag{5}$$

Proof. **Exercise!** □

Therefore the polynomial

$$f_B(y) \triangleq (y \cdot m_{b^1})(y \cdot m_{b^2}) \cdots (y \cdot m_{b^s}) \tag{6}$$

satisfies

$$A \cap B = \emptyset \implies f_B(v_A^*) \neq 0 \tag{7}$$
$$A \cap B \neq \emptyset \implies f_B(v_A^*) = 0 \tag{8}$$

Consider the matrix $(a_{ij}) = (f_i(v_j))$ where

$$f_i = f_{S_i} \quad \text{and} \quad v_j = v_{R_j}^*$$

and observe that this matrix has nonzero entries in the diagonal, and zero entries off diagonal. Therefore, it is nonsingular and thus $f_1, \ldots, f_m$ are linearly independent. To conclude the proof we show that

$$f_1, \ldots, f_m \in \operatorname{span}(g_1, \ldots, g_N)$$

with $N = \binom{r+s}{r}$ and thus $m \leq \binom{r+s}{r}$ from the linear algebra bound.
Note that f_B is polynomial in $r+1$ variables (all our vectors are in $\mathbb{R}^{r+1}$) and that (6) consists of the sum of monomials of the form

$$\lambda \cdot y_1^{k_1} y_2^{k_2} \cdots y_{r+1}^{k_{r+1}} \quad \text{where} \quad k_1 + k_2 + \cdots + k_{r+1} = s$$

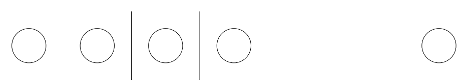
and the following claim proves $N = \binom{r+s}{r}$ and thus the theorem:

Claim 6. *The number N of integer solutions of the equation*

$$x_1 + x_2 + \cdots + x_\ell = s$$

under the condition $x_i \geq 0$ for all i is $N = \binom{s+\ell-1}{\ell-1}$

Exercise 4. *Prove the claim above. (**Hint:** Consider dividing s identical sweets among ℓ children.*[1] *One way is to lay out the sweets on a row and choose $\ell - 1$ breakpoints in between these sweets:*



In this way every child gets at least one sweet ($x_i \geq 1$). Adapt this by "borrowing" one sweets from every child to allow also $x_i \geq 0$.

A Vandermonde determinant

This is the so-called **Vandermonde** determinant:

$$\mathbf{V}_r \triangleq \det \begin{pmatrix} 1 & a_0 & a_0^2 & \ldots & a_0^r \\ 1 & a_1 & a_1^2 & \ldots & a_1^r \\ \vdots \\ 1 & a_r & a_r^2 & \ldots & a_r^r \end{pmatrix}. \tag{9}$$

which is always nonzero for distinct a_i's (proof below). Therefore any subset of $r + 1$ rows from M correspond to a matrix of the form above which is nonsingular.

Step 1: First column goes to 0 by subtracting the first row from every other row

$$\mathbf{V}_r = \det \begin{pmatrix} 1 & a_0 & a_0^2 & \ldots & a_0^r \\ 0 & a_1 - a_0 & a_1^2 - a_0^2 & \ldots & a_1^r - a_0^r \\ \vdots \\ 0 & a_r - a_0 & a_r^2 - a_0^2 & \ldots & a_r^r - a_0^r \end{pmatrix}$$

[1] This approach is described in [Juk01, Chapter 1.2].

Step 2: First row goes to 0 multiply column $i-1$ by a_0 and subtract this from the column i

$$\begin{aligned}
\mathbf{V}_r &= \det\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & a_1-a_0 & a_1(a_1-a_0) & \dots & a_1^{r-1}(a_1-a_0) \\ \vdots & & & & \\ 0 & a_r-a_0 & a_r(a_r-a_0) & \dots & a_r^{r-1}(a_r-a_0) \end{pmatrix} \\
&= \det\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & a_1 & \dots & a_1^{r-1} \\ \vdots & & & & \\ 0 & 1 & a_r & \dots & a_r^{r-1} \end{pmatrix} \cdot (a_1-a_0)(a_2-a_0)\cdots(a_r-a_0) \\
&= \underbrace{1\cdot\det\begin{pmatrix} 1 & a_1 & \dots & a_1^{r-1} \\ \vdots & & & \\ 1 & a_r & \dots & a_r^{r-1} \end{pmatrix}}_{\text{``}\mathbf{V}_{r-1}\text{''}} \cdot (a_1-a_0)(a_2-a_0)\cdots(a_r-a_0) \\
&= \cdots = \prod_{1\le i<j\le n}(a_j-a_i) \neq 0.
\end{aligned}$$

where the last inequality follows from the fact that all a_i's are distinct.

For any field $\mathbb{F}$ and any $r+1$ distinct elements $a_0, \dots, a_r \in \mathbb{F}$, the rows of the corresponding Vandermonde matrix (9) are linearly independent.

References

[Juk01] S. Jukna. *Extremal combinatorics: with applications in computer science*. Springer Verlag, 2001.

Exercises
(during next exercise class - 12.4.2018)

We shall discuss and solve together the following exercise:

Exercise 5. *Consider the following* ***twofold generalization*** *of Bollobás Theorem (see Theorem 2 in the lecture notes):*

Skew version. *If $R_1, \ldots, R_m \subseteq [n]$ are subsets of size r and $S_1, \ldots, S_m \subseteq [n]$ are subsets of size s such that*

$$R_i \cap S_i = \emptyset \qquad \text{for all } i \tag{10}$$

$$R_i \cap S_j \neq \emptyset \qquad \text{for all } i < j \tag{11}$$

then $m \leq \binom{r+s}{r}$.

Non-uniform version. *If subsets $R_1, \ldots, R_m, S_1, \ldots, S_m \subseteq [n]$ satisfy*

$$R_i \cap S_i = \emptyset \qquad \text{for all } i \tag{12}$$

$$R_i \cap S_j \neq \emptyset \qquad \text{for all } i \neq j \tag{13}$$

then

$$\sum_{i=1}^{m} \frac{1}{\binom{|S_i|+|R_i|}{|R_i|}} \leq 1. \tag{14}$$

Skew non-uniform version. *Relax (13) above as in (11), and still conclude that (14) holds.*

The first two theorems (Skew version and Non-uniform version) are true. Your task is to show that the last one is actually ***false****.*