Exercise Set 2 (Linear Algebra Methods in Combinatorics – HS11)

These exercises will be graded. Please return the solutions at the beginning of next lecture – 12.10.2011

Exercise 1 (Bipartite Oddtown – 3 Points). Suppose there are $m$ red clubs $R_1, \ldots, R_m \subseteq [n]$ and $m$ blue clubs $B_1, \ldots, B_m \subseteq [n]$ in a town of $n$ citizens. Assume that these clubs satisfy the following rules:

\begin{align*}
|R_i \cap B_i| & \quad \text{is odd for all } i \\
|R_i \cap B_j| & \quad \text{is even for all } i \neq j
\end{align*}

(13) (14)

Prove that $m \leq n$.

Exercise 2 (Bipartite Oddtown cont’d – 2 Points). Prove that clubs of the same color must be distinct (i.e., if $i \neq j$ then $R_i \neq R_j$ and $B_i \neq B_j$). Find a construction of $m = n$ for red and blue clubs satisfying the rules of Bipartite Oddtown.

Exercise 3 (Approximate Two-Distance Sets – 5 Points). In the lecture we have proven that the maximum number of points in a two-distance set in $\mathbb{R}^n$ is $m(n) \leq N$ with $N = 1 + 2n + 2n^2$. Prove the same upper bound for the following variant of the problem in which distances must be approximately equal to $\delta_1$ or to $\delta_2 > \delta_1$, according to some “error parameter” $\epsilon > 0$. The distances between the points $s_1, \ldots, s_m \in \mathbb{R}^n$ must satisfy

$$d(s_i, s_j) \in [\delta_1 - \epsilon, \delta_1 + \epsilon] \cup [\delta_2 - \epsilon, \delta_2 + \epsilon] \quad \text{for every } i \neq j$$

(15)

for some $\delta_1, \delta_2 > \delta_1$. Prove that $m \leq N$, where $N = 1 + 2n + 2n^2$, as long as $\epsilon$ is “sufficiently small”, that is, for every $\epsilon \leq \frac{\delta_1}{N}$ and $\delta_1 + \epsilon < \delta_2$ (Hint: use the result in Exercise 4 in the lecture notes, and prove that you cannot have $m = N + 1$ points.)