Exercise Set 6 (Linear Algebra Methods in Combinatorics – HS11)

These exercises will be graded. Please return the solutions at the beginning of next lecture – 9.11.2011. You can send solutions also by email (same deadline).

The first two exercises concern Lecture 5:

Exercise 1 (2 Points). Use the mod-p-RW Theorem (Theorem 5 in Lecture 5) to prove the following two results:

Reversed Oddtown Thm. If $S_1, \ldots, S_m \subseteq [n]$ satisfy

$$|S_i| = 0 \mod 2 \quad \text{for every } i$$

$$|S_i \cap S_j| \neq 0 \mod 2 \quad \text{for every } i \neq j.$$  \hspace{1cm} (6) \hspace{1cm} (7)

then $m \leq n + 1$.

Reversed mod-3-town If $S_1, \ldots, S_m \subseteq [n]$ satisfy

$$|S_i| = 0 \mod 3 \quad \text{for every } i$$

$$|S_i \cap S_j| \neq 0 \mod 3 \quad \text{for every } i \neq j.$$  \hspace{1cm} (8) \hspace{1cm} (9)

then $m \leq \binom{n}{2} + n + 1$.

Exercise 2 (3 Points). Suppose we can construct subsets $S_1, \ldots, S_m \subseteq [t]$ such that

$$|S_i| = 0 \mod 6 \quad \text{for every } i$$

$$|S_i \cap S_j| \neq 0 \mod 6 \quad \text{for every } i \neq j.$$  \hspace{1cm} (10) \hspace{1cm} (11)

Use these $m$ sets to obtain $T$-Ramsey graphs of size $m$ where $T = \binom{n}{2} + n + 2$. (Hint: now the vertices are these subsets $S_1, \ldots, S_m$ and you should color the edges according to $|S_i \cap S_j|$; note that $6 = 2 \cdot 3$ does not divide $|S_i \cap S_j|$ and use the two problems in the previous exercise.)

These two exercises concern the proof of Helly’s Theorem (Theorem 4 in the lecture notes).
Exercise 3 (3 Points). Prove the claim in (5) used in the proof of Helly’s Theorem. We restate this claim here:
Given convex subsets $C_1, \ldots, C_m$ consider the intersections
\[
I_i \triangleq \bigcap_{j \neq i} C_j \quad \text{for } i = 1, \ldots, m
\]
and assume each $I_i$ is nonempty, that is, it contains some point $p_i$. Partition these points $\{p_1, \ldots, p_m\}$ into two sets $S_A$ and $S_B$. Prove that for any $p_a \in S_A$ and $p_b \in S_B$
\[
\text{conv}(S_A) \subseteq C_b \quad \text{and} \quad \text{conv}(S_B) \subseteq C_a.
\]

Exercise 4 (2 Points). Explain the induction step to prove Helly’s Theorem. That is, suppose we have proven the theorem for $m$

(True for $m$). If $C_1, C_2, \ldots, C_m$ are convex objects in $\mathbb{R}^d$ and any $d+1$ of them intersect (in a common point), then all of them intersect (the common intersection is non-empty).

then we want to show the same for $m+1$:

(True for $m+1$). If $C_1, C_2, \ldots, C_{m+1}$ are convex objects in $\mathbb{R}^d$ and any $d+1$ of them intersect (in a common point), then all of them intersect (the common intersection is non-empty).

Show that (True for $m$) implies (True for $m+1$). Explain (a) how you reduce the number of objects from $m+1$ to $m$ and (b) why you can apply the inductive hypothesis to these new $m$ objects (prove that any $d+1$ of these new objects also intersect).