Exercise Set 7 (Linear Algebra Methods in Combinatorics – HS11)

You can submit solutions also by email by the next lecture – 16.11.2011. These exercises are non-graded but you get feedback on your submitted solutions.

Exercise 1. Let \( \mathbb{L} \) is a linear space with a non-singular scalar product and finite dimension. Prove that

\[
\dim(U) + \dim(U^\perp) = \dim(\mathbb{L}).
\]

(Hint: consider the system of equations \((u_i, x) = 0\) where \(u_i\) is the basis of \(U\).)

Exercise 2. Prove that the uniform version (Theorem 5) follows from the nonuniform Theorem 8.

Exercise 3. Prove the Proposition 10. (Hint: consider splitting the sets \(A_i\) to the uniform subfamilies with size \(\binom{n}{r}\).)

Exercise 4. Recall that a family is said critical if the following two conditions hold (see Lecture Notes 6 reloaded):

CR1: We need \(s + 1\) nodes to cover all of its members;

CR2: As soon as we remove any one member from the family, then \(s\) nodes are enough.

Bollobás uniform theorem (Theorem 5) implies that no \(r\)-uniform critical family can have more than \(\binom{s+r}{r}\) members. Prove that this result is false for \(r\)-uniform families that satisfy only CR1: show that it is possible to construct arbitrarily large \(r\)-uniform families satisfying CR1 only. (Hint: work with graphs!)

Exercise 5. Let \(\mathcal{F}\) be an \(r\)-uniform set system (a family with all members being subsets of \([n]\) of size \(r\)) of size larger than \(\binom{s+r}{r}\). Prove that there exists one \(A \in \mathcal{F}\) such that

\[
\mathcal{F}' := \mathcal{F} \setminus A \quad \text{and} \quad \mathcal{F}
\]

have the same covering number, that is, they can be both covered by \(s\) nodes, and \(s - 1\) are not enough.