1 Helly’s Theorem

The three convex objects in the figure above (left) intersect all in a common point (the intersection of the three of them is non-empty). Can you have four convex objects such that every triple of them intersects in a common point but no point belongs to all of them? It seems that if I try to add the fourth object to those above I must “get around” the intersection in the middle and thus my fourth object will not be convex (right picture).

Convexity can be defined in terms of “special” linear combinations:

*Only the last paragraph at page 7 is new and it contains the discussion made at the beginning of the following lecture (Lecture 7) and during the exercise class.
We say that \( \lambda_1 v_1 + \cdots + \lambda_m v_m \) is a \textbf{convex combination} if
\[
\lambda_1 + \cdots + \lambda_m = 1 \quad \text{and} \quad \lambda_i \geq 0
\]
(Here \( v_i \in \mathbb{R}^n \) are vectors and \( \lambda_i \in \mathbb{R} \).)

We denote the set of \textbf{all convex combinations} of vectors \( v_1, \ldots, v_m \in \mathbb{R}^n \) as
\[
\text{conv}(v_1, \ldots, v_m)
\]
where \( S = \{v_1, \ldots, v_m\} \) which is the “analogous” of the span, though restricted to convex combinations. Point and vectors are the same, so we often talk about a set of points \( S = \{v_1, \ldots, v_m\} \), and write “\( \text{conv}(S) \)” in place of “\( \text{conv}(v_1, \ldots, v_m) \)”; this is the so called \textbf{convex hull} of the set of points \( S \).

A \textbf{convex object} (or convex set) is a set \( C \subseteq \mathbb{R}^n \) which is closed under linear combinations (every convex combination of \( v_1, \ldots, v_k \in C \) belongs also to the set \( C \)).

**Theorem 1** (Helly’s Thm. – dummy version). \( \text{If } C_1, C_2, C_3, C_4 \text{ are convex objects in 2D and any three of them have non-empty intersection, then the intersection of all of them is also non-empty.} \)

\textit{Proof Idea – has a mistake.} We have four intersections (one for each “omitted” object) and each of them must contain a point (the intersections of three objects are non-empty):
\[
\begin{align*}
I_1 &:= C_2 \cap C_3 \cap C_4 \quad \to \quad p_1 \\
I_2 &:= C_1 \cap C_3 \cap C_4 \quad \to \quad p_2 \\
I_3 &:= C_1 \cap C_2 \cap C_4 \quad \to \quad p_3 \\
I_4 &:= C_1 \cap C_2 \cap C_3 \quad \to \quad p_4
\end{align*}
\]

Each segment between two points belongs to the corresponding intersections:
\[
\text{segment}(p_1, p_2) \in I_1 \cap I_2
\]

Finally, we can draw two such segments that must also intersect in some point \( p \). This common point will belong to \textbf{all} objects.
Exercise 1. Find the mistake in the proof above.

Surprisingly the same is true for any number of objects:

Theorem 2 (Helly’s Thm. in 2D). If $C_1, C_2, \ldots, C_m$ are convex objects in 2D and any three of them intersect (in a common point), then all of them intersect (the common intersection is non-empty).

Proof Idea. The proof is by induction on $m$ and it uses the fact that the intersection of two convex objects is also convex (see Exercise 2).

To get the idea we prove the case $m = 5$. We reduce to the previous theorem ($m = 4$) by replacing the last two objects (or any two of them) by their intersection:

In order to apply the theorem for $m' = 4$ we need two things:

- $C_4'$ is also convex (Exercise 2);

- Any three sets among $C_1', C_2', C_3', C_4'$ intersect
  (Exercise: use Theorem 1 to prove that, for instance, $C_1', C_2'$ and $C_4'$ intersect).

Theorem 1 says that there is a point

$$p' \in C_1' \cap C_2' \cap C_3' \cap C_4' = C_1 \cap C_2 \cap C_3 \cap C_4 \cap C_5$$

and thus the intersection of our original 5 points is nonempty.
1.1 Towards higher dimensions...

What happens in 3D? We have 5 objects and thus 5 points (coming from the intersections of all but one subset):

We can find 3 points whose convex hull intersects the line through the remaining two points.

**Lemma 3 (Radon).** Let $S$ is a set of $m \geq d + 2$ points in $\mathbb{R}^d$. Then $S$ has two disjoint subsets $S_1$ and $S_2$ whose convex hulls intersect.

**Proof.** Look at the matrix (our points are vectors in $\mathbb{R}^d$)

$$A = \begin{pmatrix} p_1 & \cdots & p_{d+2} \\ 1 & \cdots & 1 \end{pmatrix}$$

and simply because this is an $(d+1) \times (d+2)$-matrix, there is a vector $\lambda \neq 0$ such that $A\lambda = 0$. That is,

$$\lambda_1 p_1 + \cdots + \lambda_{d+2} p_{d+2} = 0 \quad \text{with} \quad \sum_i \lambda_i = 0$$

and at least one $\lambda_i \neq 0$. The partition is

$$S_1 = \{p_i|\lambda_i > 0\} \quad S_2 = \{p_i|\lambda_i < 0\}$$

(Can you tell why these subsets are nonempty?)
To visualize what happens, suppose for a moment that $S_1$ consists of the first $k$ points:

$$
\lambda_1 p_1 + \cdots + \lambda_k p_k + \lambda_{k+1} p_{k+1} + \cdots + \lambda_{d+2} p_{d+2} = 0
$$

from $S_1$

and

$$
\lambda_1 + \cdots + \lambda_k + \lambda_{k+1} + \cdots + \lambda_{d+2} = 0
$$

from $S_2$

Take $\lambda_i^+ := -\lambda_i$ for those in $S_2$ and obtain

$$
\lambda_1 p_1 + \cdots + \lambda_k p_k = \lambda_{k+1}^+ p_{k+1} + \cdots + \lambda_{d+2}^+ p_{d+2} \quad (1)
$$

$$
\sum_1 \lambda_1 + \cdots + \lambda_k = \sum_2 \lambda_{k+1}^+ + \cdots + \lambda_{d+2}^+ \quad (2)
$$

Since $\sum_1 = \sum_2$ we can obtain convex combinations by dividing both sides by this quantity:

$$
\frac{\lambda_1}{\sum_1} p_1 + \cdots + \frac{\lambda_k}{\sum_1} p_k = \frac{\lambda_{k+1}}{\sum_2} p_{k+1} + \cdots + \frac{\lambda_{d+2}}{\sum_2} p_{d+2} \quad (3)
$$

(Can you see that these are convex combinations?)

So we have a point $p$ which can be written as both a convex combination of the points in $S_1$ (left hand side of (3)) and as a convex combination of the points in $S_2$ (right hand side of (3)). That is, $p \in \text{conv}(S_1) \cap \text{conv}(S_2)$ and therefore these convex hulls intersect. (You should see that our assumption that $S_1$ consists of the first $k$ points is just for “typographical” reasons and this proof works in general – just rewrite the above steps using the summation symbol “$\sum$”, or reorder the points at the beginning so those in $S_1$ preceded those in $S_2$.)

**Theorem 4** (Helly). If $C_1, C_2, \ldots, C_m$ are convex objects in $\mathbb{R}^d$ with $m \geq d + 2$ and any $d + 1$ of them intersect (in a common point), then all of them intersect (the common intersection is non-empty).

**Proof Sketch.** We first prove the “dummy version”, that is, the case $m = d + 2$. We have the following steps:
1. Map each object $C_i$ into an intersection which “excludes $C_i$”:

$$
\begin{align*}
C_1 & \rightarrow I_1 := C_2 \cap C_3 \cap \cdots \cap C_{d+2} \rightarrow p_1 \\
C_2 & \rightarrow I_2 := C_1 \cap C_3 \cap \cdots \cap C_{d+2} \rightarrow p_2 \\
& \vdots \\
C_{d+2} & \rightarrow I_{d+2} := C_1 \cap C_2 \cap \cdots \cap C_{d+1} \rightarrow p_{d+2}
\end{align*}
$$

2. Apply Radon’s Lemma to this set of points:

$$
\text{conv}(S_1) \cap \text{conv}(S_2) \neq \emptyset
$$

which means that there is a point $p$ with

$$
p \in \text{conv}(S_1) \quad \text{and} \quad p \in \text{conv}(S_2)
$$

(4)

3. We claim that for any $p_i \in S_1$ and $p_j \in S_2$

$$
\text{conv}(S_1) \subseteq C_j \quad \text{and} \quad \text{conv}(S_2) \subseteq C_i
$$

(Exercise)

4. We can then show that $p \in C_1 \cap \cdots \cap C_m$ by combining the previous two items: for any $C_k$ either $p_k \in S_2$ or $p_k \in S_1$ which by (5) means

$$
\text{conv}(S_1) \subseteq C_k \quad \text{or} \quad \text{conv}(S_2) \subseteq C_k
$$

and thus (4) gives that $p \in C_k$ by for all $k$.

Finally the case $m > d + 2$ can be proved by induction on $m$ (Exercise).

2 “Helly-theorems” for graphs

Here is a simple example of a “combinatorial version” of Helly’s Theorem:

**Example 5.** Every three edges of a graph intersect in one node $\implies$ All edges intersect in one node.
Note that each triple can be covered by a different node, so it is not obvious that one node can cover all of them: if we have triples of edges \( T_1, T_2, \ldots \) we only know that some node \( n_1 \) covers \( T_1 \) and (a possibly different) node \( n_2 \) covers \( T_2 \), and so on...

We can see this as a vertex cover problem: (a) For every three edges one vertex is enough to cover them, implies (b) The graph has a vertex cover of size one. The following theorem generalizes the previous example (no proof for now):

**Theorem 6** (Erdős-Hajnal-Moon). If every subset of \( \binom{s+2}{2} \) edges of a graph can be covered by \( s \) nodes, then all edges of the graph can be covered by \( s \) nodes.

One can go even further and replace edges (2-subsets) by hyperedges of uniform size (\( r \)-subsets). The hypergraph is just a collection of subsets of exactly \( r \) nodes, also called \( r \)-uniform set system.\(^1\)

**Theorem 7** (Bollobás). If every subfamily of at most \( \binom{s+r}{r} \) members of an \( r \)-uniform set system can be covered by \( s \) nodes, then all members can.

We shall prove Bollobás Theorem in the next lecture. For the moment observe that these “Helly-type” theorems are of the form

\[
\text{Local Condition} \implies \text{Global Condition}
\]

**Addendum.** Now we show the “strategy” to prove the theorem. Our theorem can be “graphically” represented as

\[ \mathcal{F} \]

\[ \begin{array}{c}
\text{YES} \\
\text{YES} \\
\text{YES} \\
\end{array} \]

\[ \rightarrow \]

\[ \begin{array}{c}
\text{YES} \\
\text{YES} \\
\end{array} \]

\[ \mathcal{F} \]

\(^1\)An \( r \)-uniform set system is a family \( \mathcal{F} = \{S_1, \ldots, S_m\} \) where \( S_i \subseteq [n] \) and \( |S_i| = r \) for all \( i \). This is just a “set of subsets” and we call it family just to avoid confusion; any subset of \( \mathcal{F} \) gives another family consisting of some of the members of \( \mathcal{F} \).
where the promise is that “all small pieces” can be covered by $s$ nodes. Suppose we cannot cover our family using $s$ nodes (so we need $s + 1$ or more). We start removing, one by one, members from the family until “$s$ nodes are almost enough”:

A family is said critical if

CR1: We need $s + 1$ nodes to cover all of its members;

CR2: As soon as we remove any one member from the family, then $s$ nodes are enough.

So the idea to prove Bollobás theorem is:

- By contradiction, if we cannot cover $\mathcal{F}$ using $s$ nodes, then there is some critical subfamily ($\mathcal{F}' \subseteq \mathcal{F}$) that still cannot be covered using $s$ nodes;

\[
\begin{array}{c}
\mathcal{F} \\
\text{NO} \\
\rightarrow \\
\mathcal{F}' \\
\text{NO (critical)}
\end{array}
\]

Now observe that by hypothesis this critical $\mathcal{F}'$ must be “large”:

\[
|\mathcal{F}'| > \binom{s + r}{r}
\]  \hspace{1cm} (6)

because otherwise we can cover it with $s$ nodes (it would be one of the “small green pieces” in the picture at page 7).

- Show that every critical family must be “small”:

\[
\mathcal{F}' \text{ critical} \implies |\mathcal{F}'| \leq \binom{s + r}{r}
\]  \hspace{1cm} (7)

(This implication is due to Theorem 8 below.)

Thus we have a contradiction due to (6) and (7).
Theorem 8 (Bollobás theorem - uniform version). Let \( \{A_1, \ldots, A_m\} \) is a \( r \)-uniform set family and \( \{B_1, \ldots, B_m\} \) is a \( s \)-uniform set family such that
\[
A_i \cap B_i = \emptyset \quad \text{for all } i
\]
\[
A_i \cap B_j \neq \emptyset \quad \text{for all } i \neq j
\]
then \( m \leq \binom{r+s}{r} \).

To see that (7) follows from this theorem we observe the following:

\[ F' = \{A_1, \ldots, A_m\} \text{ critical} \implies \text{some } \{B_1, \ldots, B_m\} \text{ satisfy (8)}. \]

If we remove \( A_i \) from the critical family then

1. Some \( B_i \) of size \( s \) covers all remaining members of the family (CR1);
2. \( B_i \) cannot cover also \( A_i \) because \( s \) nodes cannot cover the whole family (CR2).

At this point you should see that Theorem 8 \( \implies \) Theorem 7.

3 Exercises

Exercise 2. Prove that the intersection of two convex objects \( C_1 \) and \( C_2 \) is also convex.

Exercise 3. Prove Theorem 2. Proceed by induction on \( m \) adapting the proof given for \( m = 5 \):

1. Reduce the number of objects to \( m' = m - 1 \);
2. Explain why you can apply the inductive hypothesis to these \( m' \) objects;
3. Derive from the inductive hypothesis that the original \( m \) objects intersect in a common point.

Also explain the base case of the induction.
Exercise 4. Prove Helly’s Theorem 4 using Radon’s Lemma 3 for the case $d = 3$ and $m = 5$ objects. That is, suppose $C_1, \ldots, C_5$ are convex objects in $\mathbb{R}^3$ and every 4 of them intersect. Show that they all intersect.

Exercise 5. Prove the result mentioned in Example 5.

More exercises are in the “Graded Set 6” – next page.
Exercise Set 6 (Linear Algebra Methods in Combinatorics – HS11)

These exercises will be graded. Please return the solutions at the beginning of next lecture – 9.11.2011. You can send solutions also by email (same deadline).

The first two exercises concern Lecture 5:

Exercise 1 (2 Points). Use the mod-p-RW Theorem (Theorem 5 in Lecture 5) to prove the following two results:

Reversed Oddtown Thm. If $S_1, \ldots, S_m \subseteq [n]$ satisfy

\begin{align*}
|S_i| &= 0 \mod 2 \quad \text{for every } i \\
|S_i \cap S_j| &\neq 0 \mod 2 \quad \text{for every } i \neq j.
\end{align*}

(9) (10)

then $m \leq n + 1$.

Reversed mod-3-town If $S_1, \ldots, S_m \subseteq [n]$ satisfy

\begin{align*}
|S_i| &= 0 \mod 3 \quad \text{for every } i \\
|S_i \cap S_j| &\neq 0 \mod 3 \quad \text{for every } i \neq j.
\end{align*}

(11) (12)

then $m \leq \binom{n}{2} + n + 1$.

Exercise 2 (3 Points). Suppose we can construct subsets $S_1, \ldots, S_m \subseteq [t]$ such that

\begin{align*}
|S_i| &= 0 \mod 6 \quad \text{for every } i \\
|S_i \cap S_j| &\neq 0 \mod 6 \quad \text{for every } i \neq j.
\end{align*}

(13) (14)

Use these $m$ sets to obtain $T$-Ramsey graphs of size $m$ where $T = \binom{n}{2} + n + 2$. (Hint: now the vertices are these subsets $S_1, \ldots, S_m$ and you should color the edges according to $|S_i \cap S_j|$; note that $6 = 2 \cdot 3$ does not divide $|S_i \cap S_j|$ and use the two problems in the previous exercise.)

These two exercises concern the proof of Helly’s Theorem (Theorem 4 in the lecture notes).
Exercise 3 (3 Points). Prove the claim in (5) used in the proof of Helly’s Theorem. We restate this claim here: Given convex subsets $C_1, \ldots, C_m$ consider the intersections

$$I_i \triangleq \bigcap_{j \neq i} C_j \quad \text{for } i = 1, \ldots, m$$

and assume each $I_i$ is nonempty, that is, it contains some point $p_i$. Partition these points $\{p_1, \ldots, p_m\}$ into two sets $S_A$ and $S_B$. Prove that for any $p_a \in S_A$ and $p_b \in S_B$

$$\text{conv}(S_A) \subseteq C_b \quad \text{and} \quad \text{conv}(S_B) \subseteq C_a.$$

Exercise 4 (2 Points). Explain the induction step to prove Helly’s Theorem. That is, suppose we have proven the theorem for $m$

(True for $m$). If $C_1, C_2, \ldots, C_m$ are convex objects in $\mathbb{R}^d$ and any $d + 1$ of them intersect (in a common point), then all of them intersect (the common intersection is non-empty).

then we want to show the same for $m + 1$:

(True for $m+1$). If $C_1, C_2, \ldots, C_{m+1}$ are convex objects in $\mathbb{R}^d$ and any $d + 1$ of them intersect (in a common point), then all of them intersect (the common intersection is non-empty).

Show that (True for $m$) implies (True for $m+1$). Explain (a) how you reduce the number of objects from $m+1$ to $m$ and (b) why you can apply the inductive hypothesis to these new $m$ objects (prove that any $d+1$ of these new objects also intersect).