Tools from the previous lectures

In the previous lecture we discussed the Helly-type geometric theorems about convex bodies and we started to discuss discrete (finite) Helly-type theorems with applications to graphs.

1 Preliminaries

Definition 1. Let $W$ is a linear space with a dimension $n$ over a field $\mathbb{F}$, $S \subset W$ is in a general position iff for all $D \subset S, |D| = n$, $D$ is linearly independent.

Curve is a map from field $\mathbb{F}$ to $\mathbb{F}^n$. For example $\alpha \in \mathbb{R} \to (f_1(\alpha), f_2(\alpha), \ldots, f_n(\alpha)) \in \mathbb{R}^n$, concretely for example $(\alpha, \sin(\alpha), \cos(\alpha)) \in \mathbb{R}^3$.

Definition 2. Let $\mathbb{F}$ is a field. The set $M_n = \{(1, \alpha, \alpha^2, \ldots, \alpha^{n-1})|\alpha \in \mathbb{F}\}$ is called a moment curve.
Moment curve has many interesting features (Carathéodory 1907, Gale 1956 etc). The important feature for us is the following:

**Proposition 3.** Let the field $\mathbb{F}$ has at least $n$ elements. The moment curve $M_n$ is in general position in $\mathbb{F}^n$.

**Proof.** Let us suppose that the field has at least $n$ different elements. The moment curve is injective i.e. it is 1-1 to its image.

We take arbitrary distinct points $x_1, \ldots, x_n$ on the moment curve, then the corresponding pre-images are $a_1, \ldots, a_n$. Consider the determinant of the following matrix

$$V_n = \det \begin{pmatrix} 1 & a_1 & a_1^2 & \ldots & a_1^{n-2} & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \ldots & a_2^{n-2} & a_2^{n-1} \\ \vdots \\ 1 & a_n & a_n^2 & \ldots & a_n^{n-2} & a_n^{n-1} \end{pmatrix}. \quad (1)$$

If the determinant is non-zero, the matrix has rank $n$ and this means the rows of the matrix (i.e. the points on the moment curve) are linearly independent.

The determinant above is called Vandermonde determinant and its value is

$$V_n = \prod_{1 \leq i < j \leq n} (a_j - a_i). \quad (2)$$

Because $a_j \neq a_i$ for any $i < j$, the determinant is non-zero and the proposition follows. The last step is to prove (2).

To compute the value of Vandermonde determinant we consider two elimination steps, in the first step all elements in the first column except the element at the position $(1, 1)$ are eliminated to zero using elementary operations which does not change the value of the determinant. The elementary operation used is to subtract the first row from the $i$-th row. We obtain

$$V_n = \det \begin{pmatrix} 1 & a_1 & a_1^2 & \ldots & a_1^{n-2} & a_1^{n-1} \\ 0 & a_2 - a_1 & a_2^2 - a_1^2 & \ldots & a_2^{n-2} - a_1^{n-2} & a_2^{n-1} - a_1^{n-1} \\ \vdots \\ 0 & a_n - a_1 & a_n^2 - a_1^2 & \ldots & a_n^{n-2} - a_1^{n-2} & a_n^{n-1} - a_1^{n-1} \end{pmatrix}. \quad (3)$$

now we eliminate the first row except the element at the position $(1, 1)$. The elementary operation used is to multiply column $i - 1$ with $a_1$ and
subtract from the column $i$. After factoring out the appropriate terms we obtain

$$V_n = \text{det} \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & a_2 - a_1 & a_2(a_2 - a_1) & \ldots & a_2^{n-3}(a_2 - a_1) & a_2^{n-2}(a_2 - a_1) \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & a_n - a_1 & a_n(a_n - a_1) & \ldots & a_n^{n-3}(a_n - a_1) & a_n^{n-2}(a_n - a_1) \end{pmatrix} =$$

$$= \prod_{i=2}^{n}(a_i - a_1)\text{det} \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 1 & a_n & \ldots & a_n^{n-3} & a_n^{n-2} \end{pmatrix} = \prod_{i=2}^{n}(a_i - a_1)V_{n-1}$$

(4)

(5)

and we can recursively compute $V_{n-1}$ obtaining the formula (2). 

The last ingredient for the proof of the main theorem enumerates a basis of homogeneous polynomials in $n$ variables and degree $k$.

**Proposition 4.** Let $P$ be the set of all homogeneous polynomials in $n$ variables $x_1, \ldots, x_n$ with degree $k$. All monic monomials $x_1^{t_1} \ldots x_n^{t_n}$ where $t_i \geq 0$ form a basis of $P$ with a size $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$.

**Proof.** Homogeneous polynomials in $n$ variables and degree $k$ have the following form $P = \left\{ \sum_{i \in I} \alpha_i x_1^{t_{i,1}} x_2^{t_{i,2}} \ldots x_n^{t_{i,n}} | t_{i,j} \geq 0, k = \sum_{j=1}^{n} t_{i,j} \right\}$. If we consider all linear combinations of different monomials $x_1^{t_1} \ldots x_n^{t_n}$ which are independent, we obtain the set $P$.

To compute the size of the set of all monic monomials denote $u_i = i + \sum_{l=1}^{i} t_l$ where $t_i \geq 0$ i.e.

$$u_1 = 1 + t_1$$
$$u_2 = 2 + t_1 + t_2$$
$$\vdots$$
$$u_{n-1} = (n-1) + t_1 + \cdots + t_{n-1}$$
$$u_n = n + t_1 + \cdots + t_n = n + k$$

(6)

Because $u_n = n + \sum_{i=1}^{n} t_i = n + k$ we obtain for $u_i$ the inequalities
1 \leq u_1 < u_2 < \cdots < u_{n-1} \leq n + k - 1. \tag{7}

There is a 1-1 correspondence between the vectors \((t_1, \ldots, t_n)\) and \((u_1, \ldots, u_{n-1})\) because in every transition from \(u_i\) to \(u_{i+1}\) a value \(1 + t_{i+1}\) is added and the difference \(u_{i+1} - u_i = 1 + t_{i+1}\). Note that the last power \(t_n\) is uniquely determined with the powers \(t_1, \ldots, t_{n-1}\) because \(k = \sum_{j=1}^{n} t_i\).

We can consider any \((n-1)\)-tuple \((u_1, \ldots, u_{n-1})\) as a subset of the set \([n+k-1]\), therefore there are \(\binom{n+k-1}{n-1}\) such \((n-1)\)-tuples.

In other words, the proposition enumerates the number of nonnegative solutions of linear Diophantine equation \(k = \sum_{j=1}^{n} t_i\).

## 2 Bollobás Theorem and Consequences

**Theorem 5** (Bollobás theorem - uniform version). Let \(\{A_1, \ldots, A_m\}\) is a \(r\)-uniform set family and \(\{B_1, \ldots, B_m\}\) is a \(s\)-uniform set family such that

\[
A_i \cap B_i = \emptyset \quad i = j
\]

\[
A_i \cap B_i \neq \emptyset \quad i \neq j
\]

then \(m \leq \binom{r+s}{s}\).

**Proof.** Let \(V = (\cup_{i=1}^{m} A_i) \cup (\cup_{i=1}^{m} B_i)\). We can suppose without loss of generality that \(V = [n]\) for some \(n\).

We associate every \(v \in V\) with a \((r+1)\)-dimensional vector from \(\mathbb{R}^{r+1}\) using a map

\[
P : v \to (p_0(v), p_1(v), \ldots, p_r(v))
\]

such that any \(r+1\) vectors \(P(v_0), P(v_1), \ldots, P(v_r)\) are linearly independent i.e. the image \(P(V)\) of a set \(V\) is in a general position.

We can achieve this using moment curve according to Proposition 3 i.e.

\[
P : v \to (1, v, \ldots, v^r).
\]

This is possible for any finite set \(V\) because the field \(\mathbb{R}\) is infinite.

Now for any \(W \subset V\) we can define a polynomial in \((r+1)\) indeterminates (variables) \(x_0, x_1, \ldots, x_r\).
\[ f_W(x) = \prod_{v \in W} (p_0(v)x_0 + p_1(v)x_1 + \cdots + p_r(v)x_r) \] (11)

which is a homogeneous polynomial with degree \(|W|\). Generally

\begin{align*}
  f_W(x) &= 0 & \text{if } x \text{ is orthogonal to any of } P(W) \\
  f_W(x) &\neq 0 & \text{if } x \text{ is not orthogonal to all of } P(W)
\end{align*}

Denote \( f_i = f_{B_i} \) and characterize \( A_i \) with a normal vector \( a_i \). The linear space \( \mathbb{R}^{r+1} \) has a standard scalar product \( x \cdot y = \sum_{i=1}^{r+1} x_i y_i \) which is non-singular, that means the only vector for which \( x \cdot x = 0 \) is 0. For linear spaces \( L \) with a non-singular scalar product, the following is true for any subspace \( U \)

\[ \dim(U) + \dim(U^\perp) = \dim(L) \] (12)

where \( U^\perp \) means the set of vectors which are orthogonal to all vectors in \( U \) (see Exercise 1).

That means in our case that there is a unique 1-dimensional subspace which is orthogonal to \( P(A_i) \) because the whole space has dimension \((r + 1)\) and \( P(A_i) \) has dimension \( r \) (the set \( A_i \) has \( r \) elements). We can choose so called normal vector \( a_i \) from this orthogonal subspace and assign it to the set \( A_i \). From (12) it follows that the vector \( a_i \) is orthogonal to any vector in the span of \( P(A_i) \) but it is not orthogonal to any other vector from \( P(V) \). Otherwise the dimension of the full space must have been \((r + 2)\) which is not true.

Specifically, \( a_i \) is not orthogonal to any vector from \( P(B_i) \) because \( A_i \) and \( B_i \) do not intersect and any vector which is not in \( A_i \) can not be orthogonal to \( a_i \) as explained above. Therefore if we use a scalar product notation for (11) we obtain

\[ f_W(x) = \prod_{v \in W} P(v) \cdot x \] (13)

For the values \( f_i(a_j) \) we have

\begin{align*}
  f_i(a_j) &\neq 0 & \text{if } i = j \quad \text{because } A_i \cap B_i = \emptyset \\
  f_i(a_j) & = 0 & \text{if } i \neq j \quad \text{because } A_i \cap B_i \neq \emptyset.
\end{align*}
which means that the matrix $f_i(a_j)$ has rank $m$ and therefore the polynomials $f_i$ are linearly independent. From linear algebra bound it follows that $m$ is smaller or equal to the dimension of the subspace which contains all polynomials $f_i$. They have $(r+1)$ variables and degree $(s)$ because the sets $B_i$ have $s$ elements. Using Proposition 4 we can conclude that $m \leq \binom{(r+1)+s-1}{(r+1)-1} = \binom{r+s}{r}$.

At the end of the last lecture we have seen that the following theorem is a consequence of uniform Bollobás theorem:

**Theorem 6.** Let $\mathcal{F}$ is a $r$-uniform, $\tau$-critical family of sets with $\tau(\mathcal{F}) = s+1$. Then $|\mathcal{F}| \leq \binom{r+s}{r}$.

Moreover, we derived from Theorem 6 a consequence which is directly applicable to graphs stating that

**Theorem 7.** Let $\mathcal{F}$ is a $r$-uniform family of sets. If each subfamily $|\mathcal{F}_1| \leq \binom{r+s}{r}$ can be covered by $s$ elements than the whole family $\mathcal{F}$ can be covered with $s$ elements.

Theorem 6 is in fact equivalent to Theorem 7. Let us prove the opposite direction Theorem 7 $\Rightarrow$ Theorem 6.

**Proof.** Suppose $\mathcal{F}$ is $r$-uniform, $\tau$-critical with $\tau(\mathcal{F}) = s + 1$. To proceed by contradiction we additionally suppose that $|\mathcal{F}| > \binom{r+s}{r}$. Denote also $l = \binom{r+s}{r}$.

We can suppose without loss of generality that $|\mathcal{F}| = \binom{r+s}{r} + 1$ because if $\mathcal{F}$ is larger say

$$\mathcal{F} = \{A_1, \ldots, A_l, A_{l+1}, \ldots, A_m\} \quad (14)$$

then we can construct a new family $\mathcal{F}_1$ setting $A_{l+1} = \bigcup_{i \geq l} A_i$. The new family is $s + 1$-critical and has $l + 1$ members.

Having $s + 1$-critical family with $l + 1$ members we can drop any member and obtain a $s$-cover for subfamily which is $l$ large. This fulfills the assumptions of Theorem 7 which means that we can cover the whole family with $s$ elements creating a contradiction with the $s + 1$-criticality of the whole family.
3 Extensions

The natural question considering the RW-theorems is whether a non-uniform versions of Helly-type theorems are possible. The response is yes but there is no linear algebra proof of this fact. Bollobás proved with combinatorial induction that

**Theorem 8.** Let \( \{A_1, \ldots, A_m\} \) and \( \{B_1, \ldots, B_m\} \) are set families such that

\[
A_i \cap B_i = \emptyset \quad i = j
\]
\[
A_i \cap B_i \neq \emptyset \quad i \neq j
\]  \hspace{1cm} (15)

then

\[
\sum_{i=1}^{m} \frac{1}{\binom{|A_i| + |B_i|}{|A_i|}} \leq 1
\]  \hspace{1cm} (16)

The uniform version (Theorem 5) follows from the stronger Theorem 8 and we leave this simple computation to Exercise 2.

On the other hand, modifying the proof of Theorem 5 it is possible to prove a stronger, so called skew version:

**Theorem 9** (Bollobás theorem - skew uniform version). Let \( \{A_1, \ldots, A_m\} \) is a \( r \)-uniform set family and \( \{B_1, \ldots, B_m\} \) is a \( s \)-uniform set family such that

\[
A_i \cap B_i = \emptyset \quad i = j
\]
\[
A_i \cap B_i \neq \emptyset \quad i < j
\]  \hspace{1cm} (17)

then \( m \leq \binom{r+s}{r} = \binom{r+s}{s} \).

**Proof.** The proof is identical to Theorem 5 only in the final stage we obtain

\[
f_i(a_j) \neq 0 \quad \text{if } i = j \quad \text{because } A_i \cap B_i = \emptyset
\]
\[
f_i(a_j) = 0 \quad \text{if } i < j \quad \text{because } A_i \cap B_i \neq \emptyset \text{ only if } i < j.
\]

leading to an upper triangular matrix instead of diagonal matrix. The consequences are the same as in Theorem 5.

There is no non-uniform version of the skew theorem as the following counter example shows.
Proposition 10 (Counter example for non-uniform skew theorem). Given \([n], n > 1\) construct the following set families. \(A_i\) will be all subsets of \([n]\) ordered decreasingly with size and \(B_i = [n] - A_i\). The families fulfill the non-uniform skew version of Theorem 8 i.e.

\[
A_i \cap B_i = \emptyset \quad i = j \\
A_i \cap B_i \neq \emptyset \quad i < j
\]

then

\[
\sum_{i=1}^{m} \frac{1}{(|A_i| + |B_i|)} \geq n
\]

Proof. Left to Exercise 3.

The example above illustrates that the theorems obtained with linear algebra bound method constitute a different development than the non-uniform theorem. Moreover, the algebraic methods allow also a development of further variations to skew theorems where the set families are replaced with subspace families.
Exercise Set 7 (Linear Algebra Methods in Combinatorics – HS11)

You can submit solutions also by email by the next lecture – 16.11.2011. These exercises are non-graded but you get feedback on your submitted solutions.

Exercise 1. Let \( \mathbb{L} \) is a linear space with a non-singular scalar product and finite dimension. Prove that

\[
\dim(U) + \dim(U^\perp) = \dim(\mathbb{L}).
\]

(Hint: consider the system of equations \((u_i, x) = 0\) where \(u_i\) is the basis of \(U\).)

Exercise 2. Prove that the uniform version (Theorem 5) follows from the nonuniform Theorem 8.

Exercise 3. Prove the Proposition 10. (Hint: consider splitting the sets \(A_i\) to the uniform subfamilies with size \(\binom{n}{r}\).)

Exercise 4. Recall that a family is said critical if the following two conditions hold (see Lecture Notes 6 reloaded):

CR1: We need \(s + 1\) nodes to cover all of its members;

CR2: As soon as we remove any one member from the family, then \(s\) nodes are enough.

Bollobás uniform theorem (Theorem 5) implies that no \(r\)-uniform critical family can have more than \(\binom{s+r}{r}\) members. Prove that this result is false for \(r\)-uniform families that satisfy only CR1: show that it is possible to construct arbitrarily large \(r\)-uniform families satisfying CR1 only. (Hint: work with graphs!)

Exercise 5. Let \(\mathcal{F}\) be an \(r\)-uniform set system (a family with all members being subsets of \([n]\) of size \(r\)) of size larger than \(\binom{s+r}{r}\). Prove that there exists one \(A \in \mathcal{F}\) such that

\[
\mathcal{F}' := \mathcal{F} \setminus A \quad \text{and} \quad \mathcal{F}
\]

have the same covering number, that is, they can be both covered by \(s\) nodes, and \(s - 1\) are not enough.