Exercise 1. Let $p$ be a prime, and $n$ and $k$ are any two positive integers. Suppose $S_1, \ldots, S_m \subseteq [n]$ satisfy

\[ |S_i| \neq 0 \mod p^k \quad \text{for every } i \quad (1) \]
\[ |S_i \cap S_j| = 0 \mod p^k \quad \text{for every } i \neq j. \quad (2) \]

Prove that $m \leq n$.

Exercise 2. Let $n$ be a positive integer. Suppose $R_1, \ldots, R_m, B_1, \ldots, B_m \subseteq [n]$ satisfy

\[ |R_i \cap B_i| \neq 0 \mod 4 \quad \text{for every } i \quad (3) \]
\[ |R_i \cap B_j| = 0 \mod 4 \quad \text{for every } i \neq j. \quad (4) \]

Prove that $m \leq n$.

Exercise 3. Let $p$ be a prime, and $n$ and $k$ are any two positive integers. Suppose $R_1, \ldots, R_m, B_1, \ldots, B_m \subseteq [n]$ satisfy

\[ |R_i \cap B_i| \neq 0 \mod p^k \quad \text{for every } i \quad (5) \]
\[ |R_i \cap B_j| = 0 \mod p^k \quad \text{for every } i \neq j. \quad (6) \]

Prove that $m \leq n$. Prove the same if the condition $6$ becomes $|R_i \cap B_j| = 0 \mod p^k$ for every $i < j$. 
Exercise 4. Let \( k_1, \ldots, k_l \) be positive integers and suppose that \( S_1, \ldots, S_m \subseteq [n] \) satisfy

\[
|S_i| \neq 0 \mod c \quad \text{for every } i \\
|S_i \cap S_j| = 0 \mod c \quad \text{for every } i \neq j
\]

where \( c = p_1^{k_1} p_2^{k_2} \cdots p_l^{k_l} \) and \( p_1, \ldots, p_l \) are prime numbers. Prove that \( m \leq h_c \cdot n \), where \( h_c \) is a constant that depends on \( c \) only.

Exercise 5. Grolmusz (2000) proved that it is possible to construct a family of \( m \geq 2^{\alpha \log^2 n / \log \log n} \) subsets \( S_1, \ldots S_m \subseteq [n] \) such that

\[
|S_i| \neq 0 \mod 6 \quad \text{for every } i \\
|S_i \cap S_j| = 0 \mod 6 \quad \text{for every } i \neq j
\]

Use his construction to obtain \( t \)-Ramsey graphs of size superpolynomial in \( t \). Explain the construction, i.e., which are the vertices, which are the edges, and prove that this is indeed a \( t \)-Ramsey graph.

Exercise 6. Suppose that \( m \) points \( s_1, \ldots, s_m \in \mathbb{R}^n \) are such that

\[
d(s_i, s_j) = 1 \quad \text{for all } i \neq j
\]

where \( d() \) is the Euclidean distance.

- Prove that \( m \leq n + 2 \).
- Improve the bound to \( m \leq n + 1 \) and show that this bound is tight.

Exercise 7. Prove that, for any \( n \) there exists \( \epsilon > 0 \) such that the following holds. If \( m \) points \( s_1, \ldots, s_m \in \mathbb{R}^n \) are such that

\[
d(s_i, s_j) \in [1 - \epsilon, 1 + \epsilon] \quad \text{for all } i \neq j
\]

where \( d() \) is the Euclidean distance, then \( m \leq O(n) \).

Exercise 8. Suppose that \( 2m \) points \( a_1, \ldots, a_m, b_1, \ldots, b_m \in \mathbb{R}^n \) are such that, for some \( \delta_1 \) and \( \delta_2 \), they satisfy

\[
d(a_i, b_j) \in \{\delta_1, \delta_2\} \quad \text{if and only if } i < j
\]

where \( d() \) is the Euclidean distance. Prove that \( m \leq O(n^2) \).
Exercise 9. Prove that for every positive integer $p$ there exists a constant $c(p)$ such that the following holds. If $m$ points $s_1, \ldots, s_m \in \mathbb{R}^n$ satisfy

$$||s_i - s_j||_p = 1 \quad \text{for all } i \neq j,$$

then $m \leq c(p)n$ where $||x||_p$ is the usual $L_p$-norm.

Exercise 10. Suppose $m$ complete bipartite graphs cover each edge of the complete graph $K_n$ an odd number of times. Provide a detailed proof that $m \geq (n-1)/2$.

Exercise 11. Consider a set of $n$ vectors $V = \{v_1, \ldots, v_n\}$, where each vector $v_i$ is in $\mathbb{R}^{r+1}$ and $n \geq 2r$. For any subset $A \subseteq [n]$ of size $r$ we define the following mapping:

$$A \mapsto V_A \triangleq \{v_i \mid i \in A\}$$

The set $V$ satisfies the following condition: For every $A$ as above there is some $v_A^* \in \mathbb{R}^{r+1}$ such that

1. The vector $v_A^*$ is orthogonal to every vector in $V_A$.
2. If $A$ and $B$ do not intersect, then $v_A^*$ is not orthogonal to any of the vectors in $V_B$ (here $B \subseteq [n]$ is also of size $r$).

Prove that $V$ must be “in general position”, that is, any subset of $r+1$ vectors from $V$ are linearly independent.

Exercise 12. We have $c$ colors. Give an explicit coloring of the complete graph with $n = \binom{t^2}{2c-1}$ nodes where each node corresponds to a subset of $[t]$ of size $2c - 1$. Color the edges $(S_i, S_j)$ according to the size of the intersection $|S_i \cap S_j|$ using at most $c$ colors so that no monochromatic component of size $t+1$ exists. Describe the coloring rule and prove the non-existence of such components.

Exercise 13. The following theorem is called Non-uniform-RW theorem.

Let $D$ be a set of $s$ integers and the subsets $S_1, \ldots, S_m \subseteq [n]$ are $D$-intersecting i.e.

$$|S_i \cap S_j| \in D \quad \text{for every } i \neq j.$$ \hfill (14)

Then the maximum number $m = m(n)$ of such subsets is $m(n) \leq \sum_{i=0}^{s} \binom{n}{i}$.
Please, adapt the proof of mod-p-RW theorem to prove the theorem above and explain why the same polynomial definition as in the proof of mod-p-RW theorem can not be used.

**Exercise 14.** The following theorem is called Non-uniform-Bolobas theorem.

Let \( \{A_1, \ldots, A_m\} \) and \( \{B_1, \ldots, B_m\} \) are set families such that

\[
A_i \cap B_i = \emptyset \quad i = j
\]

\[
A_i \cap B_i \neq \emptyset \quad i \neq j
\]

then

\[
\sum_{i=1}^{m} \frac{1}{(|A_i|+|B_i|)} \leq 1
\]

Prove that the theorem above becomes false if we change the condition 16 to \( A_i \cap B_i \neq \emptyset \ i < j \).