Exercise Set 10 – HS12
(Linear Algebra Methods in Combinatorics)

These exercises will be graded. Please return the solutions at the beginning of next lecture – 5.12.2012. You can send solutions also by email (same deadline).

Two exercises on circuit complexity.

Exercise 1 (2 Points). In this exercise we consider these depth-2 “XOR-of-OR” circuits:

\[
\begin{array}{c}
\begin{array}{c}
\lor \\
\vdots \\
\lor \\
\oplus
\end{array}
\end{array}
\]

where each of the OR gates is connected to a subset of the input variables (no variable is negated) and the output is the XOR of all these gates. We say that such a circuit computes a graph \( G = (V, E) \) if \( G \) has \( n \) nodes and its edges satisfy

\[
E = \{(u, v) : C(e_{u,v}) = 1\}
\]

where

\[
e_{u,v} = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0)
\]

is the input vector in which only variables \( x_u \) and \( x_v \) are set to 1.

Prove that any such XOR-of-OR circuit computing a graph \( G \) must have size \( s + 1 \geq \text{rank}_{\mathbb{F}_2}(A_G) \), where \( A_G \) is the adjacency matrix of \( G \) (recall that \( s \) is the number of OR gates).

**Hint:** Use the result of Exercise 4 in Lecture 9.

Exercise 2 (3 Points). Explicit constructions of bipartite Ramsey graphs (see below) are also difficult to obtain. A probabilistic argument shows that such graphs can be obtained from a (suitable) subset \( S \subseteq \{0,1\}^k \) as follows: Take two copies of \( S \) as the two sets of vertices of the bipartite graph
and add an edge between \( u \) (on the left side) and \( v \) (on the right side) if and only if \( \langle u, v \rangle = 1 \) over \( \mathbb{F}_2 \). There is an \( S \) for which this graph \( G_S \) has \( n = \Omega(2^{k/2}) \) nodes and the complete bipartite graph \( K_{k,k} \) is not contained in the graph \( G_S \) nor in its bipartite complement\(^2\) – the graph \( G_S \) is said bipartite Ramsey in this case.

Prove that there exist bipartite Ramsey graphs that can be computed by a depth-2 XOR-of-OR circuit of size logarithmic in the number of nodes of the graph (describe the circuit and prove this statement).

**Note:** The formal definition of “bipartite Ramsey” is not crucial for this exercise.

Three exercises on “Euclidean coloring” problem in this lecture.

**Exercise 3 (1 Point).** Consider the following strategy to prove a lower bound on the number of colors \( m(n) \) needed to color \( \mathbb{R}^n \). There exists some unit-distance graph \( G \) in \( \mathbb{R}^n \) which has a large clique, say \( K \), and therefore \( m(n) \geq \chi(G) \geq |K| \). Prove that the best lower bound that is possible to obtain with this approach is \( m(n) \geq n + 1 \).

**Exercise 4 (1 Point).** Suppose we can cover the whole space \( \mathbb{R}^n \) using a set of open balls, \( \mathcal{B} = \{ B_1, B_2, \ldots, B_i, \ldots \} \), of radius \( 1/2 \). We construct a graph \( G \) whose vertices are the centers of these balls, \( \mathcal{C} = \{ c_1, c_2, \ldots, c_i, \ldots \} \), and an edge \( (c_i, c_j) \) exists if and only if the distance \( d(c_i, c_j) \leq 2 \). Prove that, if you can color \( G \) with \( c \) colors, then you can color the whole space \( \mathbb{R}^n \) with \( c \) colors (i.e., \( m(n) \leq \chi(G) \)).

**Hint:** Extend the coloring of the centers to the whole space. Prove that in this way two points \( p \) and \( p' \) at distance 1 cannot get the same color.

\(^2\)For any bipartite graph \( G = (L \cup R, E) \), its bipartite complement is the bipartite graph \( \bar{G} = (L \cup R, \bar{E}) \) where \( \bar{E} = \{(u, v) \notin E | u \in L \land v \in R \} \).
**Exercise 5 (1 Point).** Show that the graph $G$ in Exercise 4 can be colored with $9^n$ colors if the set $C = \{c_1, c_2, \ldots, c_i, \ldots\}$ satisfies $d(c_i, c_j) \geq 1/2$ for all distinct $c_i, c_j \in C$.

**Hint:** Use the fact that a ball of radius $9/4$ cannot contain more than $9^n$ balls of radius $1/4$. 