Exercise Set 2 – HS12
(Linear Algebra Methods in Combinatorics)

These exercises will be graded and each exercise counts 1 point. Please return the solutions at the beginning of next lecture – 10.10.2012 – or send them by email to paolo.penna@inf.ethz.ch (same deadline).

Exercise 1 (Skew Bipartite Oddtown). Suppose there are \( m \) red clubs \( R_1, \ldots, R_m \subseteq [n] \) and \( m \) blue clubs \( B_1, \ldots, B_m \subseteq [n] \) in a town of \( n \) citizens. Assume that these clubs satisfy the following rules:

\[
| R_i \cap B_i | \quad \text{is odd for all } i \quad (16)
\]

\[
| R_i \cap B_j | \quad \text{is even for all } i > j \quad (17)
\]

Prove that (1) \( m \leq n \), (2) clubs of the same color must be distinct (i.e., if \( i \neq j \) then \( R_i \neq R_j \) and \( B_i \neq B_j \)), and (3) find a construction of \( m = n \) for red and blue clubs satisfying the rules (16)-(17).

Exercise 2 (Mod-c-Town). Consider a composite number \( c = p_1^{k_1} p_2^{k_2} \cdots p_l^{k_l} \), where \( p_1, \ldots, p_l \) are prime numbers and \( k_1, \ldots, k_l \) are nonnegative integers. Suppose that \( S_1, \ldots, S_m \subseteq [n] \) satisfy

\[
| S_i | \neq 0 \mod c \quad \text{for every } i \quad (18)
\]

\[
| S_i \cap S_j | = 0 \mod c \quad \text{for every } i \neq j \quad (19)
\]

Prove that \( m \leq 1 \cdot n \).

Exercise 3 (Approximate Two-Distance Sets). In the lecture we have proven that the maximum number of points in a two-distance set in \( \mathbb{R}^n \) is \( m(n) \leq N \) with \( N = 1 + 2n + 2n^2 \). Prove the same upper bound for the following variant of the problem in which distances must be approximately equal to \( \delta_1 \) or to \( \delta_2 > \delta_1 \), according to some “error parameter” \( \epsilon > 0 \). The distances between the points \( s_1, \ldots, s_m \in \mathbb{R}^n \) must satisfy

\[
d(s_i, s_j) \in [\delta_1 - \epsilon, \delta_1 + \epsilon] \cup [\delta_2 - \epsilon, \delta_2 + \epsilon] \quad \text{for every } i \neq j \quad (20)
\]

for some \( \delta_1, \delta_2 > \delta_1 \). Prove that \( m \leq N \), where \( N = 1 + 2n + 2n^2 \), as long as \( \epsilon \) is “sufficiently small”. That is, find an explicit \( \epsilon_0 > 0 \), which depends on \( \delta_1, \delta_2 \) and \( n \), such that \( m \leq N \) for all \( \epsilon < \epsilon_0 \). (Note: to make calculations simpler, assume \( d() \) to be the square of the Euclidean distance.)
Exercise 4 (One-Distance Sets). A regular simplex is a set of \( n + 1 \) points in \( \mathbb{R}^n \) such that any two points are at distance 1. Prove that no set with this property can have more points. That is, for any \( n \), if \( m \) points \( s_1, \ldots, s_m \in \mathbb{R}^n \) are such that

\[
d(s_i, s_j) = 1 \quad \text{for all } i \neq j
\]

where \( d() \) is the Euclidean distance, then \( m \leq n + 1 \). If you cannot prove this bound, try to prove the simpler bound \( m \leq n + 2 \).

Exercise 5 (One-Distance Sets with \( L_p \)-Norm). Prove that for every positive even integer \( p \) there exists a constant \( c(p) \) such that the following holds. If \( m \) points \( s_1, \ldots, s_m \in \mathbb{R}^n \) satisfy

\[
||s_i - s_j||_p = 1 \quad \text{for all } i \neq j,
\]

then \( m \leq n^{c(p)} \) where \( ||x||_p = (\sum_k |x_k|^p)^{1/p} \) is the usual \( L_p \)-norm.