Exercise Set 4 – HS12
(Linear Algebra Methods in Combinatorics)

These exercises will be graded. Please return the solutions at the beginning of next lecture – 24.10.2012. You can send solutions also by email (same deadline).

Exercise 1 (1 point). Consider the variant of the Jigsaw Puzzle with graphs in which we require every edge to be covered an odd number of times (instead of exactly once). Let \( m(n) \) denote the minimum number of complete bipartite subgraphs that are necessary in this variant. Prove that \( m(n) \geq \frac{n-1}{2} \).

Exercise 2 (1 point). Suppose \( S_1, \ldots, S_m \subseteq \{1, \ldots, n\} \) satisfy the conditions of Fisher inequality:

\[
|S_i \cap S_j| = c \quad \text{for all } i \neq j
\]

Show that if one subset \( S_k \) has size \( |S_k| = c \) then it must be \( m \leq n \).

Exercise 3 (3 points). Write an alternative proof of Fisher inequality based on the following notion:

A matrix \( A \) is said positive semidefinite if

\[
x^T Ax \geq 0 \quad \text{for all } x \neq 0
\]

and it is said positive definite if

\[
x^T Ax > 0 \quad \text{for all } x \neq 0
\]

Show that the matrix \( A = (a_{ij}) \) with \( a_{ij} = v_i \cdot v_j \) is positive definite and explain how from this you obtain Fisher inequality.

Note: The vectors \( v_i \)'s are the usual incident vectors of the subsets. As in the proof in the notes you can assume that each subset has size strictly larger than \( c \), i.e., \( |S_i| > c \) for all \( i \).
Exercise 4 (2 points). Extend the idea of the cubic construction (Section 2 in the lecture notes) to obtain the following: A construction of a 3-coloring of the complete graph with \( n = \binom{t}{3} \) nodes so that there is no monochromatic complete subgraph of size \( t + 1 \).

**Note:** By “3-coloring” we mean that we color the edges using colors red, blue, and green.

Exercise 5 (2 points). It is possible to extend the Ramsey Theorem for two colors in the lecture notes as follows:

**Theorem.** For every natural numbers \( c \) and \( t \), there exists a natural number \( n = R(t; c) \) such that, if we color the complete graph with \( n \) or more nodes using \( c \) colors, then there must be monochromatic complete subgraph of size \( t \).

Show that the Ramsey Theorem for \( c \) colors (theorem above) implies the following:

**Theorem.** For every \( c \) there exists a natural number \( n = S(c) \) such that, if we color the integers \( \{1, 2, \ldots, n\} \) using \( c \) colors, then there exist integers \( a, b \) and \( a + b \) that get the same color.

**Hint:** Look at what happens if we color the edges of a complete graph so that the color of every edge \( (u, v) \) depends only on the difference \( u - v \) (consider the vertices as integers). Note that the theorem does not assume \( a \neq b \).