Exercise 1. Construct a critical set for $r = 2$ and $s = 2$ matching the bound of Bollobás theorem.

Exercise 2. Generalize the proof of Bollobás Theorem to show this “skew version” of it

Theorem 2. If $R_1, \ldots, R_m \subseteq [n]$ are subsets of size $r$ and $S_1, \ldots, S_m \subseteq [n]$ are subsets of size $s$ such that

\begin{align*}
R_i \cap S_i &= \emptyset \quad \text{for all } i \\
R_i \cap S_j &\neq \emptyset \quad \text{for all } i < j
\end{align*} \tag{10} \tag{11}

then $m \leq \binom{r+s}{r}$.

Exercise 3. Let $F = F_q$ with $q = 2^k$. Construct a set of $2^k$ vectors in $F^4$ such that any four of them are linearly independent.

Exercise 4. Reconsider Claim 5 in the proof of Bollobás Theorem, when instead of the matrix $M$ in (3) we use an arbitrary $n \times (r+1)$ matrix $M'$. Which part of Claim 5 still hold and which does not?

Exercise 5. Recall that a family is said critical if the following two conditions hold (see lecture notes):

CR1: We need $s + 1$ nodes to cover all of its members;

CR2: As soon as we remove any one member from the family, then $s$ nodes are enough.

Bollobás uniform theorem (Theorem 3) implies that no $r$-uniform critical family can have more than $\binom{s+r}{r}$ members. Prove that this result is false for $r$-uniform families that satisfy only CR1: show that it is possible to construct an arbitrarily large $r$-uniform family satisfying CR1 only and such that $s + 1$ nodes are enough to cover all members of the family.
Exercise 6. Let $\mathcal{F}$ be an $r$-uniform set system of size larger than \( s^r \). Prove that there exists one $A \in \mathcal{F}$ such that
\[
\mathcal{F}' := \mathcal{F} \setminus A \quad \text{and} \quad \mathcal{F}
\]
have the same covering number, that is, they can be both covered by $s$ nodes, and $s - 1$ are not enough.