The next three exercises are on the proof of Razborov lower bound.

**Exercise 1.** We know that for every $m$ there is a polynomial $p_\land$ which differs from the AND of $m$ bits on at most $2^m / 2^r$ of the possible 0/1-inputs. Consider the following “local procedure” to approximate a given circuit $C$ with a low-degree polynomial. For each AND gate consider its “direct inputs in the circuit” (left figure below):

```
\begin{array}{c}
y_1' \quad \cdots \quad y_m' \\
\land \\
\downarrow \\
C(x_1, \ldots, x_n)
\end{array}
```

Replace each gate by the polynomial $p_\land$ on the “direct inputs” of the gate (see right figure above). Show that there is a circuit $\hat{C}$ for which the resulting polynomial $\hat{C}$ differs from $C$ on more than $|C| 2^m / 2^r$ inputs.

**Exercise 2.** Prove Lemma 4 in the lecture notes by adapting the idea for the special case $p = x_1 \cdots x_m$.

**Exercise 3.** Derive Lemma 3 from Lemma 4 (see lecture notes).

The next two exercises on depth-2 circuits will help for future exercises.

**Exercise 4.** Consider this type of depth-2 circuits:
where each of the OR gates is connected to a subset of the input variables (no variable is negated) and the output is the XOR of all these gates. For any \( u \) and \( v \), let \( e_{u,v} \) denote the input in which only variables \( x_u \) and \( x_v \) are set to 1:

\[
ed_{u,v} = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0)
\]

Consider the graph over \( n \) nodes whose edges are

\[
E = \{(u, v) : C(e_{u,v}) = 1\}
\]

Show that

\[
(u, v) \in E \iff s - |D_u \cap D_v| = 1 \mod 2
\]

where \( s \) is the number of OR gates and \( D_u \) is the set of OR gates that is not connected to variable \( x_u \).

**Exercise 5.** Also in this exercise we restrict our attention to depth-2 circuits \( C \) of the type described in Exercise 4 ("XOR-of-OR circuits"). Consider the following \( n \times n \) bipartite graph \( G = (V_1 \cup V_2, E) \). The vertices of each side \( V_i \) correspond to all 0/1-vectors of length \( k \) (so \( n = |V_1| = |V_2| = 2^k \)). Two vertices \( u \in V_1 \) and \( v \in V_2 \) are adjacent if and only if \( \langle u, v \rangle = 1 \) over \( \mathbb{F}_2 \). Show that there is a depth-2 “XOR-of-OR” circuit \( C \) that computes the edges of this graph, that is

\[
E = \{(u, v) : C(e_{u,v}) = 1\}
\]

and that uses \( O(\log n) \) OR gates.