Tools from the previous lectures

Recall the construction of Ramsey graphs based on **Oddtown** and **Fisher** inequality:

\[ |S_i \cap S_j| \in \{0, 1, 2\} \]

1 One trick: Multilinearization

If we restrict to “0/1 variables” then these two polynomials are identical:

\[ x_1^2 x_3^5 + x_2^3 \quad \text{and} \quad x_1 x_3 + x_2 \]

because \( x_i^k = x_i \) for \( x_i \in \{0, 1\} \). The second polynomial is **multilinear**, meaning that every monomial is the product of distinct variables.
For these polynomials we can get good bounds:

The number of multilinear polynomials linearly independent is at most

\[
\binom{n}{d} + \binom{n}{d-1} + \cdots + \binom{n}{0}
\]

where \(d\) is the maximum degree and \(n\) is the number of variables.

**Exercise 1.** Prove (1) using the linear algebra bound.

**Exercise 2.** Prove that for \(3d \leq n + 1\)

\[
\binom{n}{d} + \binom{n}{d-1} + \cdots + \binom{n}{0} < 2^{\binom{n}{d}}
\]

## 2 Restricted intersection theorems

**Ray-Chaudhuri – Wilson Theorem (very informal)** You can’t have many sets if the size of their pairwise intersections assume few values.

This theorem “extends Oddtown”:

**Theorem 1** (mod-p-RW). Suppose the subsets \(S_1, \ldots, S_m \subseteq [n]\) satisfy

\[
|S_i| \not\equiv D \pmod{p} \quad \text{for every } i
\]

\[
|S_i \cap S_j| \equiv D \pmod{p} \quad \text{for every } i \neq j
\]

where \(D = \{\delta_1, \ldots, \delta_d\}\). Then the maximum number \(m = m(n)\) of such subsets is

\[
m(n) \leq \binom{n}{d} + \binom{n}{d-1} + \cdots + \binom{n}{0}.
\]

This theorem “extends Fisher inequality”:

**Theorem 2** (uniform-RW). Suppose distinct subsets \(S_1, \ldots, S_m \subseteq [n]\) satisfy

\[
|S_i| = s \quad \text{for every } i
\]

\[
|S_i \cap S_j| \in D \quad \text{for every } i \neq j
\]

where \(D = \{\delta_1, \ldots, \delta_d\}\) and \(s\) is a fixed constant. Then the maximum number \(m = m(n)\) of such subsets is

\[
m(n) \leq \binom{n}{d} + \binom{n}{d-1} + \cdots + \binom{n}{0}.
\]
2.1 Proof of uniform-RW

The uniform RW Theorem follows from the mod-$p$-RW Theorem

\[
\text{mod-}p\text{-RW} \implies \text{uniform-RW}
\]

Two basic observations give us the proof

1. We can always assume $s \not\in D$ (Exercise)

2. For $p > s$ we have $|S_i \cap S_j| \in D \iff |S_i \cap S_j| \in D \pmod p$

So every $S_1, \ldots, S_m \subseteq [n]$ satisfying the conditions of uniform RW Theorem must also satisfy the conditions of mod-$p$-RW Theorem.

2.2 Proof of mod-$p$-RW

An approach similar to oddtown and two-distance sets
(linear algebra bound with polynomials)

We can represent subsets by 0/1 vectors (as in Oddtown) and the inner product gives us the size of the intersections:

\[
|S_i \cap S_j| = v_i \cdot v_j
\]

The following polynomial tells us whether the size of the intersections is in $D = \{\delta_1, \ldots, \delta_d\}$:

\[
F(x, y) \triangleq (x \cdot y - \delta_1)(x \cdot y - \delta_2) \cdots (x \cdot y - \delta_d).
\]

Observe that for $f_i(x) \triangleq F(x, v_i)$ we have

\[
\begin{align*}
    f_i(v_j) &= 0 \pmod p & \text{for every } i \neq j \\
    f_i(v_i) &\neq 0 \pmod p & \text{for every } i
\end{align*}
\]

Therefore

- $f_1, \ldots, f_m$ are linearly independent
- $f_i : \{0, 1\}^n \to \mathbb{R}$ is a polynomial of degree at most $d$

We can make these polynomials multilinear and therefore the bound in (1) yields $m \leq \binom{n}{d} + \binom{n}{d-1} + \cdots + \binom{n}{0}$.
3 Superpolynomial Ramsey graphs

We generalize the cubic construction (see figure at page 1) as follows:

\[ |S_i \cap S_j| \in \{0, 1, 2, \ldots, p-2, p-1, p, p+1, \ldots, 2p-2, 2p-1, 2p, 2p+1, \ldots, (k-1)p-1, (k-1)p, \ldots, kp-2\} \]

where the subsets have all the same size \( s = kp - 1 \). This exercise tells us how to pick \( k \):

**Exercise 3.** Show that the graph cannot contain a red clique of size \( 2^{\binom{n}{2}} \) nor a blue clique of size \( t = 2^{\binom{n}{k-1}} \).

By picking the “right” \( k \) we get this:

|An explicit construction of \( t \)-Ramsey graphs \( G_p = (V_p, E_p) \) with
|---
|\(|V_p| = \binom{n}{p^2-1}\) and \( t = 2^{\binom{n}{p-1}}\) |

We can optimize \( n \) observing that for \( n = p^3 \) the following holds (Exercise): for every \( \epsilon > 0 \) there exists \( t_\epsilon \) such that

\[ |V_p| \geq t^{(1-\epsilon)\ln t/\ln \ln t} \quad (9) \]

for all \( t \geq t_\epsilon \). This leads to the following (Exercise 5 below):

**Superpolynomial explicit construction**

For every \( \epsilon > 0 \) there is an explicit construction of \( t \)-Ramsey graphs of size at least \( t^{(1-\epsilon)\ln t/\ln \ln t} \).
Exercise 1. Give some lower bound for (some special case of) the mod-p-RW-Theorem. (Describe a set $D = \{\delta_1, \ldots, \delta_d\}$ and some $p$ for which a lower bound $m \geq \binom{n}{d}$ exists – show a construction. Can you find such an example with $|D| > 1$?)

Exercise 2. Prove that the upper bound we obtain for the mod-p-RW Theorem is (in general) not the best possible. (Describe a set $D = \{\delta_1, \ldots, \delta_d\}$ and some $p$ for which it is possible to prove an upper bound $m \leq UB$ with $UB < \binom{n}{d} + \binom{n}{d-1} + \cdots$)

Exercise 3. Prove the following “nonuniform-RW Theorem”:

**Theorem.** If $S_1, \ldots, S_m \subseteq [n]$ are distinct subsets satisfying

$$|S_i \cap S_j| \in \{\delta_1, \ldots, \delta_d\} \quad \text{for every } i \neq j \quad (10)$$

then $m \leq \binom{n}{d} + \binom{n}{d-1} + \cdots + \binom{n}{0}$.

Exercise 4. Adapt to construction of superpolynomial ramsey graphs in the lecture notes to obtain a coloring of a the complete graph graph with $N = \binom{n}{kp}$ nodes such that (a) The largest blue clique has size at most $2\binom{n}{k}$ and (b) the largest red clique has size at most $2\binom{n}{p-1}$.

(Note: Here $p$ is a prime, $k$ is any positive integer, and $n$ is an integer $n \geq \max\{3k, 3p\}$ so that the approximation in (2) in the lecture notes applies.)

Exercise 5. Complete the analysis of the superpolynomial construction of Ramsey graphs: (a) Prove (9) using estimate of the binomial coefficient and (b) Then show how (9) implies a construction of $t$-Ramsey graphs of size at least $t^{(1-c)\ln t/\ln \ln t}$, also when $t$ is not of the form $2^{(p^3)}$.

(Hint: Use the Prime Number Theorem saying that for every positive integer $c$ there is a prime $p \leq c$ with $p \geq c/\ln c$.)