1 Helly’s Theorem

The three convex objects in the figure above (left) intersect all in a common point (the intersection of the three of them is non-empty). Can you have four convex objects such that every triple of them intersects in a common point but no point belongs to all of them? The red object in the right picture is not convex.

Convexity can be defined in terms of “special” linear combinations:

We say that $\lambda_1 v_1 + \cdots + \lambda_m v_m$ is a convex combination if

$$\lambda_1 + \cdots + \lambda_m = 1 \quad \text{and} \quad \lambda_i \geq 0$$

(Here $v_i \in \mathbb{R}^n$ are vectors and $\lambda_i \in \mathbb{R}$.)
We denote the set of all convex combinations of vectors \( v_1, \ldots, v_m \in \mathbb{R}^n \) as
\[
\text{conv}(v_1, \ldots, v_m)
\]
which is the “analogous” of the span, though restricted to convex combinations. Point and vectors are the same, so we often talk about a set of points \( S = \{v_1, \ldots, v_m\} \), and write “\( \text{conv}(S) \)” in place of “\( \text{conv}(v_1, \ldots, v_m) \)”; this is the so called convex hull of the set of points \( S \).

A convex object (or convex set) is a set \( C \subseteq \mathbb{R}^n \) which is closed under linear combinations (every convex combination of \( v_1, \ldots, v_k \in C \) belongs also to the set \( C \)).

**Theorem 1** (Helly’s Thm. – dummy version). If \( C_1, C_2, C_3, C_4 \) are convex objects in 2D and any three of them have non-empty intersection, then the intersection of all of them is also non-empty.

*Proof Idea – has a mistake.* We have four intersections (one for each “omitted” object) and each of them must contain a point (the intersections of three objects are non-empty):

\[
\begin{align*}
I_1 &:= C_2 \cap C_3 \cap C_4 \quad \to \quad p_1 \\
I_2 &:= C_1 \cap C_3 \cap C_4 \quad \to \quad p_2 \\
I_3 &:= C_1 \cap C_2 \cap C_4 \quad \to \quad p_3 \\
I_4 &:= C_1 \cap C_2 \cap C_3 \quad \to \quad p_4
\end{align*}
\]

Each segment between two points belongs to the corresponding intersections:

\[
\text{segment}(p_1, p_2) \in I_1 \cap I_2
\]

Finally, we can draw two such segments that must also intersect in some point \( p \). This common point will belong to all objects:
Exercise 1. Find the mistake in the proof above.

Surprisingly the same is true for any number of objects:

**Theorem 2** (Helly’s Thm. in 2D). If $C_1, C_2, \ldots, C_m$ are convex objects in 2D and any three of them intersect (in a common point), then all of them intersect (the common intersection is non-empty).

**Proof Idea.** The proof is by induction on $m$ and it uses the fact that the intersection of two convex objects is also convex (see Exercise 2).

To get the idea we prove the case $m = 5$. We reduce to the previous theorem ($m = 4$) by replacing the last two objects (or any two of them) by their intersection:

$$
\begin{array}{cccc}
C_1 & C_2 & C_3 & C_4 & C_5 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
C_1' & C_2' & C_3' & C_4' & C_5' \\
& \downarrow & \downarrow & \downarrow & \text{C_4' \cap C_5'}
\end{array}
$$

In order to apply the theorem for $m' = 4$ we need two things:

- $C_4'$ is also convex (Exercise 2);
- Any three sets among $C_1', C_2', C_3', C_4'$ intersect (Exercise: use Theorem 1 to prove that, for instance, $C_1', C_2', C_4'$ intersect).

Theorem 1 says that there is a point

$$p' \in C_1' \cap C_2' \cap C_3' \cap C_4' = C_1 \cap C_2 \cap C_3 \cap C_4 \cap C_5$$

and thus the intersection of our original 5 points is nonempty.

1.1 Towards higher dimensions...

What happens in 3D? We have 5 objects and thus 5 points (coming from the intersections of all but one subset):
We can find 3 point whose convex hull intersects the line through the remaining two points.

**Lemma 3** (Radon). Let \( S \) is a set of \( m \geq d + 2 \) points in \( \mathbb{R}^d \). Then \( S \) has two disjoint subsets \( S_A \) and \( S_B \) whose convex hulls intersect.

**Proof.** Look at the matrix (our points are vectors in \( \mathbb{R}^d \))

\[
M = \begin{pmatrix}
p_1 & \cdots & p_{d+2} \\
1 & \cdots & 1
\end{pmatrix}
\]

and simply because this is an \((d+1) \times (d+2)\)-matrix, there is a vector \( \lambda \neq 0 \) such that \( M \lambda = 0 \). That is,

\[
\lambda_1 p_1 + \cdots + \lambda_{d+2} p_{d+2} = 0 \quad \text{with} \quad \sum_{i} \lambda_i = 0
\]

and at least one \( \lambda_i \neq 0 \). The partition is

\[
S_A = \{ p_a | \lambda_a > 0 \} \quad S_B = \{ p_b | \lambda_b < 0 \}
\]

(Can you tell why these subsets are nonempty?)

So we have

\[
\sum_{a \in A} \lambda_a p_a + \sum_{b \in B} \lambda_b p_b = 0
\]

and

\[
\sum_{a \in A} \lambda_a + \sum_{b \in B} \lambda_b = 0
\]
Take $\lambda^+_b \triangleq -\lambda_b$ for those in $S_B$ and obtain

$$\sum_{a \in A} \lambda_a p_a = \sum_{b \in B} \lambda^+_b p_b$$

and

$$\sum_{a \in A} \lambda_a = \sum_{b \in B} \lambda^+_b$$

Since $\text{sum}_A = \text{sum}_B$ we can obtain convex combinations by dividing both sides by this quantity:

$$\sum_{a \in A} \frac{\lambda_a}{\text{sum}_A} p_a = \sum_{b \in B} \frac{\lambda^+_b}{\text{sum}_B} p_b$$

(1)

(Can you see that these are convex combinations?)

This tells us that there is a point $p \in \text{conv}(S_A) \cap \text{conv}(S_B)$ and thus these convex hulls intersect.

**Theorem 4** (Helly). If $C_1, C_2, \ldots, C_m$ are convex objects in $\mathbb{R}^d$ with $m \geq d + 2$ and any $d + 1$ of them intersect (in a common point), then all of them intersect (the common intersection is non-empty).

**Proof Sketch.** We first prove the case $m = d + 2$ (“dummy version”). We have the following steps:

1. Map each object $C_i$ into an intersection which “excludes $C_i$”:

   $C_1 \rightarrow I_1 := C_2 \cap C_3 \cap \cdots \cap C_{d+2} \rightarrow p_1$

   $C_2 \rightarrow I_2 := C_1 \cap C_3 \cap \cdots \cap C_{d+2} \rightarrow p_2$

   $\vdots$

   $C_{d+2} \rightarrow I_{d+2} := C_1 \cap C_2 \cap \cdots \cap C_{d+1} \rightarrow p_{d+2}$

2. Apply Radon’s Lemma to this set of points:

   $$\text{conv}(S_A) \cap \text{conv}(S_B) \neq \emptyset$$

   which means that there is a point $p$ with

   $$p \in \text{conv}(S_A) \quad \text{and} \quad p \in \text{conv}(S_B)$$

   (2)
3. We claim that for any \( p_a \in S_A \) and \( p_b \in S_B \)
\[
\text{conv}(S_a) \subseteq C_b \quad \text{and} \quad \text{conv}(S_b) \subseteq C_a
\] (3)
(Exercise)

4. We can then show that \( p \in C_1 \cap \cdots \cap C_m \) by combining the previous two items: for any \( C_k \) either \( p_k \in S_2 \) or \( p_k \in S_1 \) which by (3) means
\[
\text{conv}(S_1) \subseteq C_k \quad \text{or} \quad \text{conv}(S_2) \subseteq C_k
\]
and thus (2) gives that \( p \in C_k \) by for all \( k \).

Finally the case \( m > d + 2 \) can be proved by induction on \( m \) (Exercise). \( \square \)

2 \textbf{“Helly-theorems” for graphs}

Here is a simple example of a “combinatorial version” of Helly’s Theorem:

**Example 5.** Every three edges of a graph intersect in one node \( \implies \) All edges intersect in one node.

Note that each triple can be covered by a different node, so it is not obvious that one node can cover all of them: if we have triples of edges
\[ T_1, T_2, \ldots \]
we only know that some node \( n_1 \) covers \( T_1 \) and (a possibly different) node \( n_2 \) covers \( T_2 \), and so on...

We can see this as a vertex cover problem: (a) For every three edges one vertex is enough to cover them, implies (b) The graph has a vertex cover of size one. The following theorem generalizes the previous example (no proof for now):

**Theorem 6** (Erdős-Hajnal-Moon). If every subset of \( \binom{s+2}{2} \) edges of a graph can be covered by \( s \) nodes, then all edges of the graph can be covered by \( s \) nodes.

One can go even further and replace edges (2-subsets) by hyperedges of uniform size (r-subsets). The hypergraph is just a collection of subsets of exactly \( r \) nodes, also called \textbf{r-uniform set system}.\(^1\)

\(^1\)An \( r \)-uniform set system is a family \( \mathcal{F} = \{S_1, \ldots, S_m\} \) where \( S_i \subseteq [n] \) and \( |S_i| = r \) for all \( i \). This is just a “set of subsets” and we call it family just to avoid confusion; any subset of \( \mathcal{F} \) gives another family consisting of some of the members of \( \mathcal{F} \).
Theorem 7 (Bollobás). If every subfamily of at most \((s+r)^r\) members of an \(r\)-uniform set system can be covered by \(s\) nodes, then all members can.

We shall prove Bollobás Theorem in the next lecture. For the moment observe that these “Helly-type” theorems are of the form

\[
\text{Local Condition} \implies \text{Global Condition}
\]

### 3 Exercises

**Exercise 2.** Prove that the intersection of two convex objects \(C_1\) and \(C_2\) is also convex.

**Exercise 3.** Prove Theorem 2. Proceed by induction on \(m\) adapting the proof given for \(m = 5\):

1. Reduce the number of objects to \(m' = m - 1\);
2. Explain why you can apply the inductive hypothesis to these \(m'\) objects;
3. Derive from the inductive hypothesis that the original \(m\) objects intersect in a common point.

Also explain the base case of the induction.

**Exercise 4.** Prove Helly’s Theorem 4 using Radon’s Lemma 3 for the case \(d = 3\) and \(m = 5\) objects. That is, suppose \(C_1, \ldots, C_5\) are convex objects in \(\mathbb{R}^3\) and every 4 of them intersect. Show that they all intersect.

**Exercise 5.** Prove the result mentioned in Example 5.

More exercises are in the “Graded Set 6” – next page.
Exercise Set 6 – HS12
(Linear Algebra Methods in Combinatorics)

These exercises will be graded. Please return the solutions at the beginning of next lecture – 7.11.2011. You can send solutions also by email (same deadline).

Two exercises on constructions of Ramsey graphs (Lecture 5):

**Exercise 6 (3 Points).** We say that a graph $G = (V, E)$ can be represented by a set $\mathcal{P}$ of polynomials (in $n$ variables over the field $\mathbb{F}$) if to every vertex $v$ we can assign a polynomial $p_v \in \mathcal{P}$ and $s_v \in \mathbb{F}^n$ such that this holds:

\[
\begin{align*}
\text{For all } v \in V & \quad p_v(s_v) \neq 0 \quad \text{(4)} \\
\text{If } (u, v) \in E \text{ then } & \quad p_v(s_u) = 0 \quad \text{(5)}
\end{align*}
\]

Prove that $G$ does not contain a clique of size more than $k$ if

\[ \mathcal{P} \subseteq \text{span}(f_1, \ldots, f_k). \]

Apply this result to the superpolynomial construction of Ramsey graphs in the previous lecture: What are the polynomials representing the graph $G_{\text{red}}$ of all red edges (resp., $G_{\text{blue}}$ of all blue edges)? Over which fields are these polynomials?

**Exercise 1 (2 Points).** Suppose that it is possible to construct a family of $m \geq n^{\alpha \ln n/\ln \ln n}$ subsets $S_1, \ldots, S_m \subseteq [n]$ such that

\[
\begin{align*}
|S_i| \equiv 0 \pmod{6} & \quad \text{for every } i \quad \text{(6)} \\
|S_i \cap S_j| \neq 0 \pmod{6} & \quad \text{for every } i \neq j. \quad \text{(7)}
\end{align*}
\]

($\alpha > 0$ is some constant)

Use this construction to obtain $t$-Ramsey graphs of size superpolynomial in $t$. Explain the construction, i.e., which are the vertices, which are the edges, and prove that this is indeed a $t$-Ramsey graph.

Two exercises on the proof of Helly’s Theorem:

**Exercise 2 (2 Points).** Find the mistake in the proof of Theorem 1 in the lecture notes and suggest how to “adjust” the proof of this theorem without using Radon’s Lemma.
Exercise 3 (1 Point). Prove the claim in (3) used in the proof of Helly’s Theorem. We restate this claim here: Given convex subsets $C_1, \ldots, C_m$ consider the intersections 

$$I_i \triangleq \bigcap_{j \neq i} C_j \quad \text{for } i = 1, \ldots, m$$

and assume each $I_i$ is nonempty, that is, it contains some point $p_i$. Partition these points $\{p_1, \ldots, p_m\}$ into two sets $S_A$ and $S_B$. Prove that for any $p_a \in S_A$ and $p_b \in S_B$

$$\text{conv}(S_A) \subseteq C_b \quad \text{and} \quad \text{conv}(S_B) \subseteq C_a.$$ 

Last exercise on another result about convex sets:

Exercise 4 (3 Points). Let $S = \{s_1, \ldots, s_m\} \subseteq \mathbb{R}^d$ be a set of points. Show that any set in the convex hull of $S$ can be expressed as a convex combination of just $d + 1$ points of $S$. That is, for every $p \in \text{conv}(S)$, there are $d + 1$ points $s'_1, \ldots, s'_{d+1} \in S$ such that $p \in \text{conv}(s'_1, \ldots, s'_{d+1})$.

**Hint:** Given any $d + 2$ points/vectors $v_1, \ldots, v_{d+2}$ the points/vectors

$$(v_2 - v_1), (v_3 - v_1), \ldots, (v_{d+2} - v_1)$$

are linearly dependent (simply because these are $d + 1$ vectors of length $d$), so you can write $0$ as nontrivial combination of these points. The trivial identity $p = p + 0$ is also useful.