1 Warm up

Which of these games have pure Nash equilibria (PNE)?

<table>
<thead>
<tr>
<th></th>
<th>H</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>1, -1</td>
<td>-1, 1</td>
</tr>
<tr>
<td>T</td>
<td>-1, 1</td>
<td>1, -1</td>
</tr>
</tbody>
</table>

Matching Pennies

<table>
<thead>
<tr>
<th></th>
<th>B</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>2, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td>S</td>
<td>0, 0</td>
<td>1, 2</td>
</tr>
</tbody>
</table>

Battle of Sexes

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>-4, -4</td>
<td>-1, -5</td>
</tr>
<tr>
<td>S</td>
<td>-5, -1</td>
<td>-2, -2</td>
</tr>
</tbody>
</table>

Prisoners’ Dilemma

*(Synchronous) Best-Response Dynamics:* Players play their best response infinitely many times, one by one in a fixed order (round robin).

What happens for the three games above?

**Example 1** Two nodes, 1 and 2, want to send traffic to another destination node \( d \). Their strategy is to choose the next hop the traffic is sent to (one of the neighbors). The following picture shows the physical network and the preferences of each node (which path to use) near the corresponding node:

*The material of this lecture is taken from [NSVZ11] where you can find several other applications of best-response mechanisms. There you have a more precise, extensive, and formal description of best-response mechanisms, plus further pointers into the literature.*
Each node prefers to reach \( d \) via the other node, but if they both send their own traffic to each other they fail (which is the least preferable option for both).

**Question:** What happens if the two nodes move (play) always simultaneously? What happens if node 1 plays “1 \( \rightarrow \) 2” at each step (while the other node plays best-response)?

**Best Response:**

1. No convergence in *asynchronous* settings.
2. Not incentive compatible.

For which games this does not happen?

**Asynchronous Best-Response Dynamics:** At each step an adversary activates an arbitrary subset of players who best respond to the current profile (the adversary also chooses a starting strategy profile). The adversary must activate each player an infinite number of times.

The choice of the adversary and the “response strategies” of each player determine an infinite sequence

\[ s^0 \implies s^1 \implies \cdots s^t \implies \cdots \]

If the game converges (after finitely many steps \( T \) we have \( s^T = s^{T+1} = s^{T+2} = \cdots \)) then the utility of each player \( i \) is \( u_i(s^T) \). If the game keeps “oscillating” then we consider an upper bound on what the player can get (the worst case for us and the best for the player) that is \( \limsup_{t \to \infty} u_i(s^t) \).
Base game $G \implies$ Repeated game $G^*$

\[
s_i \in S_i \quad \text{response strategy } R_i() \in S_i
\]

\[
u_i(s) \quad \text{total utility } \Gamma_i := \limsup_{t \to \infty} u_i(s^t)
\]

**Definition 2** Best-response are **incentive compatible** for $G$ if repeated best-responding is a Nash equilibrium for the repeated game $G^*$, that is, for every $i$

\[
\Gamma_i \geq \Gamma'_i
\]

where $\Gamma_i$ is the total utility when all players best respond and $\Gamma'_i$ is the total utility when all but $i$ best respond (starting from the same initial profile $s^0$ and applying the same activation sequence).

## 2 “Nice” Games

Consider this game (with a unique PNE):

<table>
<thead>
<tr>
<th></th>
<th>Player 1</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 2</td>
<td>$A$</td>
<td>$B$</td>
</tr>
<tr>
<td>$A$</td>
<td>2,1</td>
<td>0,0</td>
</tr>
<tr>
<td>$B$</td>
<td>3,0</td>
<td>1,2</td>
</tr>
</tbody>
</table>

Best response works as follows

\[
(A, A) \xrightarrow{\text{Player}^1} (B, A) \xrightarrow{\text{Player}^2} (B, B) \xrightarrow{\text{Player}^1} (B, B) \xrightarrow{\text{Player}^2} (B, B) \cdots \xrightarrow{} (B, B)
\]

Player 1 improves if he/she does not best response (keep playing $A$):

\[
(A, A) \xrightarrow{\text{Player}^1} (A, A) \xrightarrow{\text{Player}^2} (A, A) \xrightarrow{\text{Player}^1} (A, A) \xrightarrow{} \cdots \xrightarrow{} (A, A)
\]

| Convergence but no incentive compatibility |

**Exercise 1** For the following game

\[
(A, A) \xrightarrow{\text{Player}^1} (A, A) \xrightarrow{\text{Player}^2} (A, A) \xrightarrow{\text{Player}^1} (A, A) \xrightarrow{} \cdots \xrightarrow{} (A, A)
\]
find best response strategies that never converge (keep oscillating between different profiles). Find other best response strategies for which we do have convergence. **Hint:** You may start by considering the adversary that activates players in round-robin fashion: \(1, 2, 1, 2, \ldots\)

Two intuitions/ideas:

1. Introduce tie breaking rule.
2. Eliminate “useless” strategies.

### 2.1 Convergence

Look at this three player game (we show only the payoffs\(^1\) of Player 3 for strategy \(A\), for strategy \(B\), and for strategy \(C\)):

\[
\begin{array}{c|cc}
\text{L} & \text{R} \\
\hline
\text{u} & 2 & 0 \\
\text{d} & 0 & 0 \\
\end{array}
\quad
\begin{array}{c|cc}
\text{L} & \text{R} \\
\hline
\text{u} & 0 & 0 \\
\text{d} & 0 & 2 \\
\end{array}
\quad
\begin{array}{c|cc}
\text{L} & \text{R} \\
\hline
\text{u} & 1 & 0 \\
\text{d} & 0 & 1 \\
\end{array}
\]

\(A\) \quad \(B\) \quad \(C\)

You can show that (Exercise)

1. Neither of these strategies is dominant.
2. Neither of these strategies is (weakly) dominated.
3. Strategy \(C\) satisfies the following definition:

**Definition 3 (never best response (NBR))**  
A strategy \(s_i \in S_i\) is a never best response (for tie breaking rule \(\prec\)) if there is always another strategy that gives a better payoff or that gives the same payoff but is better w.r.t. to this tie breaking rule: for all \(s_{-i}\), there exists \(s_i' \in S_i\) such that one of these holds

---

\(^1\)We use the term utility and payoff interchangeably.
1. \( u_i(s_i, s_{-i}) < u_i(s_i', s_{-i}) \) or
2. \( u_i(s_i, s_{-i}) = u_i(s_i', s_{-i}) \) and \( s_i \prec_i s_i' \).

This condition is enough to guarantee convergence (of best response):

**Definition 4 (NBR-solvable)** A game \( G \) is NBR-solvable if iteratively eliminating NBR strategies results in a game with one strategy per player. That is, there exists a tie breaking rule \( \prec \), sequence \( p_1, \ldots, p_\ell \) of players, and a corresponding sequence of subsets of strategies \( E_1, \ldots, E_\ell \) such that:

1. Initially \( G_0 = G \) and \( G_i + 1 \) is the game obtained from \( G_i \) by removing the strategies \( E_i \) of player \( p_i \);
2. Strategies \( E_i \) are NBR for \( \prec \) in the game \( G_{i-1} \).
3. The final game \( G_{\ell+1} \) has one strategy for each player (this unique profile is thus a PNE for \( G \)).

A sequence of players and of strategies as above is called an elimination sequence for the game \( G \).

**Exercise 2** Consider the 3-player game described at the beginning of this section where the payoffs of Player 1 and Player 2 are either the payoffs of Prisoners Dilemma or those of Matching Pennies depending on the strategy chosen by Player 3:

- \( A \) or \( B \) \( \implies \) Prisoners Dilemma;
- \( C \) \( \implies \) Matching Pennies

For instance, if Player 3 chooses \( A \) then the payoffs are

\[
\begin{array}{c|cc}
 & L & R \\
\hline
u & -4, -4, 2 & -1, -5, 0 \\
d & -5, -1, 0 & -2, -2, 0 \\
\end{array}
\]

Show that this game is NBR-solvable (show the tie breaking rule \( \prec \) and the elimination sequence).
Lemma 5 (rounds vs subgames) Let $p_1, \ldots, p_{\ell}$ be the players of any elimination sequence for the game under consideration. Suppose that players $p_1, \ldots, p_k$ always best respond (according to the prescribed tie breaking rule $\prec$). Then, for any initial profile and for any activation sequence, every profile after the $k$-th round is a profile in the subgame $G_{k+1}$.

Before proving the lemma we observe that it implies convergence:

Theorem 6 (convergence) For NBR-solvable games best response (according to the prescribed tie breaking rule $\prec$) converge even in the asynchronous case.

Proof. Take $k = \ell$ and observe that $G_{\ell+1}$ contains only one profile. □ □

Proof of Lemma 5. Denote by $\text{round}_j$ the last time step of the $j$-th round in the activation sequence. Obviously for any $t$ we have $s^t \in G_0 = G$. Now consider $t \geq \text{round}_1$ and observe that, since player $p_1$ has been activated at least once the corresponding strategy satisfies $^2$

$$s^t_{p_1} \notin E_1$$

which is equivalent to $s^t \in G_1$ for all $t \geq \text{round}_1$.

To prove the analogous for player $p_2$ we observe that, in the 2-nd round player $p_2$ is activated and, since $s^t \in G_1$ and since $p_2$ plays best response, for $t \geq \text{round}_2$ we have $s^t_{p_2} \notin E_2$. Since we have previously proved $s^t_{p_1} \notin E_1$, this implies $s^t \in G_2$ for $t \geq \text{round}_2$.

We can then continue and prove, by induction, that after the $k$-th round player $p_k$ does not play any strategy in $E_k$ and thus $s^t \in G_k$ for all $t \geq \text{round}_k$. □ □

2.2 Incentive Compatible

Look (again) at this game:

$^2$More in detail, if the player is activated at time $t'$ then at time $t'+1$ his/her profile is not in $E_1$; If the player is not activated at time $t'$ then her strategy at time $t'+1$ remains the same.
What’s bad (for incentive compatibility): The unique PNE does not give Player 1 the highest possible payoff he/she can get in this game.

Definition 7 (NBR-solvable with clear outcome) A NBR-solvable game $G$ has a clear outcome if the following holds. Let $s^*$ be the unique profile of the game $G_{\ell+1}$ (a PNE of $G$). There exists an elimination sequence consisting of players $p_1, \ldots, p_\alpha, \ldots, p_\ell$ and strategies $E_1, \ldots, E_\alpha, \ldots, E_\ell$ (according to Definition 4) such that the following holds for every player $i$. If $a$ is the first occurrence of $i$ in the sequence (i.e. $p_a = i$ and $p_1 \neq i, \ldots, p_{a-1} \neq i$) then, in the corresponding game $G_{a-1}$ the PNE $s^*$ is globally optimal for $i$, that is, for every other profile $\hat{s}$ in the game $G_{a-1}$ it holds that $u_i(\hat{s}) \leq u_i(s^*)$.\footnote{The original definition in [NSVZ11] is more general, but here we use this slightly weaker one since it is enough for the applications we study.}

Theorem 8 (incentive compatibility) For NBR-solvable games best response (according to the prescribed tie breaking rule $\prec$) are also incentive compatible.

Proof. Compare the case in which all players best respond to the case in which player $i$ does not best respond (while the others best respond). In particular, we consider the two sequences of profiles

All best respond: $s^0 \Rightarrow s^1 \Rightarrow s^2 \Rightarrow \cdots \Rightarrow s^* \Rightarrow s^* \cdots$

All but $i$ best respond: $s^0 \Rightarrow \hat{s}^1 \Rightarrow \hat{s}^2 \Rightarrow \cdots \Rightarrow \hat{s}^t \Rightarrow \hat{s}^{t+1} \cdots$

We want to show that starting from some finite $T$ the utility of $i$ in the second sequence is not better than the “final” utility in the first sequence:

$$u_i(\hat{s}^t) \leq u_i(s^*) \quad \text{for all } t \geq T \quad (1)$$

This implies $\hat{\Gamma}_i \leq \Gamma_i$ that is the incentive compatibility condition (see Definition 2). Consider the elimination sequence specific for player $i$ (see Definition 7):

<table>
<thead>
<tr>
<th>Player:</th>
<th>$p_1$</th>
<th>$\cdots$</th>
<th>$p_{k-1}$</th>
<th>$i$</th>
<th>$p_{k+1}$</th>
<th>$\cdots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NBR Strategies:</td>
<td>$E_1$</td>
<td>$\cdots$</td>
<td>$E_{k-1}$</td>
<td>$E_i$</td>
<td>$E_{k+1}$</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>Current Game:</td>
<td>$G_0$</td>
<td>$\cdots$</td>
<td>$G_{k-2}$</td>
<td>$G_{k-1}$</td>
<td>$G_k$</td>
<td>$\cdots$</td>
</tr>
</tbody>
</table>
We know from Lemma 5 that after round $k - 1$ the profile must be in the game $G_{k-1}$.\textsuperscript{4} Since the PNE $s^*$ is globally optimal for $i$ in this game, we have $u_i(s^t) \leq u_i(s^*)$ for all $t \geq \text{round}_{k-1}$. This proves Inequality (1) and thus the theorem.

\section{BGP Games}

Several Autonomous Systems are connected to each other:

![BGP Diagram]

The Border Gateway Protocol (BGP) specifies how to forward traffic. Each node in this graph chooses neighbor ("next hop"):\textsuperscript{4}

\textsuperscript{4}Since $i$ does not appear in the elimination sequence before position $k$, all players $p_1, \ldots, p_{k-1}$ are different from $i$ and thus they all play best response.
3.1 BGP “in Theory”...

BGP game (static version)
1. Players = Nodes
2. Strategies = Neighbors
3. Strategy profile = Set of paths (or loops)
4. Utilities = Order over the paths connecting $i$ to $d$

\[ P_1 \prec_i P_2 \prec_i \cdots \prec_i P_k \]

and any path $\emptyset$ which does not connect $i$ to $d$ is strictly worse:

\[ \emptyset \prec_i P_1. \]

Consider this instance:
There is no PNE.

**Dispute Wheel:** every node prefers routing over the next one in the "wheel"

with preferences

\[ Q_i \prec_{w_i} R_i Q_{i+1} \]

no convergence + no incentive compatible
3.2 ...BGP “in Practice”

<table>
<thead>
<tr>
<th>Gao-Rexford Model</th>
<th>⇒ No Dispute Wheel</th>
<th>⇒ BPG Converges</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Incentive Compatible</td>
</tr>
</tbody>
</table>

There are two types of **commercial relationships** between ASs:

- peer
- peer
- customer
- provider

Each node $i$ classifies paths according to its commercial relationship with the neighbor in the path (first hop): 
1. **customer paths**, 
2. **peer paths**, and 
3. **provider paths**:

The top path is a customer path because the first hop is from $i$ to a customer of $i$. Similarly, we have peer and provider paths (all neighbors of $i$ can be grouped into these three classes). The preferences of each node $i$ respect this classification:

<table>
<thead>
<tr>
<th>Gao-Rexford model (first version):</th>
</tr>
</thead>
<tbody>
<tr>
<td>(GR1) provider paths ≺ peer paths ≺ customer paths</td>
</tr>
</tbody>
</table>

Dispute wheel is still possible:

```
 12d     21d
 1_2     2_1
 1d_ 2d
∅_∅_∅
```
Gao-Rexford model (second version):
(GR1) provider paths ≺ peer paths ≺ customer paths
(GR2) transit traffic to/from my customers only

Consider this path:

\[ \text{traffic from } i \]

\[ i \longrightarrow j \longrightarrow k \longrightarrow d \]

It may happen that node \( j \) does not allow transit traffic from node \( i \):
- Node \( j \) chooses \( k \) as its next hop, but
- Node \( j \) does not forward the traffic coming from \( i \) to node \( j \)

There are only two cases where a node \( j \) allows transit traffic:

1. "from my customer"

2. "to my customer"

In all other cases a node \( j \) does not allow transit traffic:
If node $j$ does not allow transit traffic from node $i$ then any path $P = i \rightarrow j \rightarrow \cdots d$ represents a “failure” for $i$ which we denote with the symbol $\emptyset$. Such “failing” path have always the lowest utility 0.

**Example 9** Reconsider our previous example with all nodes having “peer-to-peer” relationships:

Now we consider the path “12d” as a **failure for node** 1 because its traffic will not be forwarded by node 2, though node 2 is forwarding its own traffic.
to $d$. Therefore the preferences of node 1 must be as shown in the picture. A similar argument holds for the path “21d” with respect to node 2.

**Exercise 3** Show that the following dispute wheel is still possible:

![Dispute Wheel Diagram]

that is, conditions GR 1 and GR 2 hold.

---

**Gao-Rexford model (final version):**

(1) $\emptyset \prec$ provider paths $\prec$ peer paths $\prec$ customer paths

(2) transit traffic to/from my customers only

(3) no customer-provider cycles

(3) says that no AS is indirectly a provider of itself.

(1) can be rewritten in terms of utilities as

$$0 = u_i(\emptyset) < u_i($$provider-path$) < u_i($peer-path$) < u_i($customer-path$)$

for any provider-path, any peer-path and any customer-path of $i$.

### 3.3 Gao-Rexford $\implies$ No Dispute Wheel

See Section A.1 and related exercises.

### 3.4 No Dispute Wheel $\implies$ NBR-solvable with clear outcome

The key idea to construct an appropriate elimination sequence is to identify what we call “happy paths”:  

---

14
A path $h_1 \rightarrow h_2 \rightarrow \cdots \rightarrow h_l \rightarrow d$ in a subgame $G_i$ is an **happy path** if this path gives the highest possible payoff to all of these nodes:

$$h_a \rightarrow h_{a+1} \rightarrow \cdots \rightarrow h_l \rightarrow d$$

is $h_a$'s top ranked path among those that are available in the subgame $G_i$.

To see the idea of how happy paths give an elimination sequence:

The elimination sequence goes **“from right to left”**:

15
1) \( h \) eliminates all strategies other than \( h \to d \) from the current subgame \( G_i \) and this gives us \( G_{i+1} \). In this subgame \( G_{i+1} \) it is still true that the path is an happy path and thus \( h_{i-1} \) can eliminate all strategies other than \( h_{i-1} \to h_i \). We can continue until the first node in the happy path has eliminated all but the \( \text{\textquotedbl}h_1 \to h_2\text{\textquotedbl} \) strategy.

2) In the resulting subgame we find another happy path and repeat the previous step until there are no happy paths that start with a node with at least two strategies.

Suppose at the end of this process we included all nodes:

| Every node belongs to some happy path. | (2) |

Then the final subgame consists of a game with one strategy per player. At each step we eliminate strategies that give the node a non-optimal payoff in the current subgame. So the starting game is NBR-solvable with clear outcome.

### 3.4.1 No Dispute Wheel \( \Rightarrow \) Condition (2)

We show that if there is no happy path then there must be a Dispute Wheel. Given that there is no happy path, starting from a node \( w_0 \) its top ranked path is not an happy path:

\[
TR_{w_0} = w_0 \rightarrow i_1 \rightarrow \cdots \rightarrow w_1 \rightarrow i_a \rightarrow \cdots \rightarrow i_l \rightarrow d
\]

and \( w_1 \) is the rightmost node (closest to \( d \)) for which the subpath

\[
w_1 \rightarrow i_a \rightarrow \cdots \rightarrow i_l \rightarrow d
\]

is not \( w_1 \)'s top ranked available path which is instead

\[
TR_{w_1} = w_1 \rightarrow j_1 \rightarrow \cdots \rightarrow w_2 \rightarrow j_{a'} \rightarrow \cdots \rightarrow j_{l'} \rightarrow d
\]

where \( w_2 \) is (again) the rightmost node in this path for which the corresponding subpath is not top ranked for it (this because there is no happy path). Since there is no happy path this can go on until we get some \( w_k \) such that

\[
TR_{w_k} = w_1 \rightarrow n_1 \rightarrow \cdots \rightarrow w_{k+1} \rightarrow n_{a''} \rightarrow \cdots \rightarrow n_{l''} \rightarrow d
\]

and \( w_{k+1} \) is one of the previously considered \( w_j \)'s. For instance, if \( w_{k+1} = w_0 \) then we get the Dispute Wheel.
by setting $R_i Q_{i+1} := TR_{w_i}$. If $w_{k+1} = w_s$ then we get a smaller Dispute Wheel with nodes $w_s, w_{s+1}, \ldots, w_k$.

BGP “in Practice” (Gao-Rexford model):

YES convergence + YES incentive compatible

References

A Omitted parts

A.1 Gao-Rexford $\implies$ No Dispute Wheel

We show that the network cannot contain nodes and paths that form a dispute wheel. We prove the result only for these simpler wheels (paths $P_i$ and $Q_i$ consist of a single link):

Recall that $\emptyset$ denotes any path that does not allow $w_i$ to reach $d$ (in particular if $w_{i+1}$ does not allow transit traffic from $w_i$) and the utility is $u_{w_i}(\emptyset) = 0$. This and the preferences of the nodes

$Q_i \prec_{w_i} R_i Q_{i+1}$

imply that $w_{i+1}$ must allow transit traffic from $w_i$. This is possible only in one of these two cases (GR2):

Exercise: show that in either case we must have a dispute wheel.