There is a definition of the \( O \)-Notation that is different from the one given at the lecture. Namely, for a function \( g : \mathbb{N} \to \mathbb{R}^+ \), let

\[
O(g) := \{ f : \mathbb{N} \to \mathbb{R}^+ \mid \exists c > 0, n_0 \in \mathbb{N} \forall n \geq n_0 : f(n) \leq cg(n) \}.
\]

Analogously, we say that a function \( f \) grows asymptotically at least as much as \( g \), if \( f \in \Omega(g) \) with

\[
\Omega(g) := \{ f : \mathbb{N} \to \mathbb{R}^+ \mid \exists c > 0, n_0 \in \mathbb{N} \forall n \geq n_0 : f(n) \geq cg(n) \}.
\]

A function \( f \) grows asymptotically like \( g \) when \( f \in O(g) \) and \( f \in \Omega(g) \). We will write this as \( f = \Theta(g) \).

For these exercises, you can choose to use the definition given at the lecture, or use the above definition.

**Exercise 1.1** The Set \( \Theta(g) \).

Give a definition of the set \( \Theta(g) \) as compactly as possible (i.e., with the fewest possible parameters and quantifiers), analogously to the above definitions for the sets \( O(g) \) and \( \Omega(g) \).

**Exercise 1.2** Proofs about \( O \)-Notation.

Prove or disprove the following statements, where \( f, g, h : \mathbb{N} \to \mathbb{R}^+ \).

a) If \( f \in \Omega(g) \) and \( g \in \Omega(f) \), then \( f = g \).

b) If \( f \in \Theta(g) \), then also \( g \in \Theta(f) \).

c) If \( f \leq g \), then \( f \in O(g) \).

d) If \( f \in O(g) \), then \( f \leq g \).

e) For every \( a, b \in \mathbb{N}, a \leq b \) it holds that \( \sqrt{n} \in \Theta(\sqrt{n}) \).

f) For every \( a, b \in \mathbb{N}, a \leq b \) it holds that \( n^a + n^b \in \Theta(n^b) \).

g) If \( f = O(g) \) and \( f = O(h) \), then \( O(g) = O(h) \).

**Exercise 1.3** Asymptotic growth of functions I.

Sort the following functions from left to right such that: if function \( f \) is on the left of \( g \), then \( f \in O(g) \). For example: the functions \( n^3, n^7, n^9 \) are already in the right order, since \( n^3 \in O(n^7) \) and \( n^7 \in O(n^9) \).

\[
\log(n^n), \sqrt{2n}, \left(\frac{n}{2}\right)^2, 2^n, \frac{1}{n}, \log(n), n!, n^3, \sqrt{n}
\]

**Exercise 1.4** Asymptotic growth of functions II.

Indicate which of the following statements are correct and which are false, and justify your decision.

a) \( k^n \in O(2^n) \) for \( k, n \in \mathbb{N}, k > 2 \)

b) \( n^k \in O(2^n) \) for \( k \in \mathbb{N} \)

c) \( \log_a(n) \in O(\log_b(n)) \) for \( a, b \in \mathbb{N}, a, b > 1 \)

d) \( n! \in O(n^n) \)

e) \( \frac{n}{\log n} \in \Theta(n) \)

**Hand-in:** until Wednesday 29th February 2012.