Algorithmic Game Theory HS12
Graded exercise sheet 11

Regulations:

- There will be a total of four graded exercise sheets during this semester, this is one of them.
- Your solutions to these exercise sheets will be graded. The three highest out of your four achieved grades will account for 10% of your final grade for the course each (so 30% of the grade in total).
- You are expected to solve them carefully and then write a nice complete exposition of your solution using LaTeX. The appearance of your solution will also be part of the grade. (In particular this means that you are expected to use a spell-checker!)
- You are welcome to discuss the tasks with your fellow students, but we expect each of you to hand in your own individual write-up.
- The deadline for handing in the solution is on Monday December 17, until 15:15. Please send your PDF via email to matus.mihalak@inf.ethz.ch, or sandro.montanari@inf.ethz.ch. Your file should have the name <Surname>.pdf, where <Surname> should be exchanged by your family name.

EXERCISE 11.1: (5 points)
Consider an auction with \( k \) identical goods and \( n \) players. Each player wants to have any of these goods, and exactly one. The value of player \( i \), \( i = 1, 2, \ldots, n \), for winning (i.e., getting one of the goods) is \( v_i \), a private information of the player.

Decide, whether the following auctions for the described situation are truthful:

a) Run \( k \) Vickrey auctions (also known as the second-price auction), one after another, for one single good.

b) Ask the players for their valuations, receive bids \( b_i \) from the players (expressing their valuation), sort the players according to the bids (highest bid first), and assign the first \( k \) bidders one good each.
   i) Charge each winner \( i \) (one of the first \( k \) bidders) the \((i + 1)\)-th bid (in the sorted order).
   ii) Charge each winner \( i \) (one of the first \( k \) bidders) the \((k + 1)\)-th bid (in the sorted order).

EXERCISE 11.2: (5 points)
In this exercise we consider combinatorial auctions with single-minded bidders. Recall that in such an auction every player is only interested in getting the goods in \( S_i \subseteq U \) (where \( U \) is the set of goods). The player \( i \) values this bundle \( S_i \) with \( v_i \in \mathbb{R}^+ \). Both \( S_i \) and \( v_i \) are the private information of player \( i \). Every player \( i \) submits a bid \((B_i, b_i)\) to the auction, expressing the desire to get the bundle \( B_i \) and that the player values it with \( b_i \).
Consider the following modification of the LOS mechanism:

a) the outcome (i.e., the decision of the mechanism about which player is granted its bundle) remains unchanged;

b) the price that any winner $i$ pays is $\sqrt{|B_i|} \cdot \frac{b_i}{\sqrt{|B_j|}}$, where $j > i$ is the first $j$ after $i$ (in the order given by the descending values of $b_k/\sqrt{|B_k|}$, $k = 1, \ldots, n$) for which $B_i \cap B_j \neq \emptyset$. The payment will be zero if no such $j$ exists.

Is this mechanism truthful?

**EXERCISE 11.3:** (4 points)
Recall the problem of scheduling $m$ jobs on $n$ machines, where every job $j$ has a load (size) $l_j$, and every machine $i$ needs $t_i$ time to process one unit of load. The machines are the players and $t_i$ is the private information of player $i$. Every player $i$ submits to the mechanism value $b_i$ with which it claims that the machine $i$ needs time $b_i$ to process one unit of load. The mechanism then assigns to every machine $i$ a set of jobs $J_i$ such that $J_1, J_2, \ldots, J_n$ forms a partition of the jobs $\{1, 2, \ldots, m\}$, and decides for every player $i$ the amount of money $p_i$ the player $i$ gets. The load (or work) of machine $i$ in this assignment is $W_i(b, b_{-i}) = \sum_{j \in J_i} l_j$. The expression $t_i \cdot W_i(b, b_{-i})$ is the cost to machine $i$.

Consider the following greedy strategy for assigning jobs to machines: Sort the jobs such that $l_j \geq l_{j+1}$; go through the jobs in the resulting order, and assign iteratively job $j$ to a machine $i$ as specified in the following: Let $W_i^{(j-1)}$ denote the load of machine $i$ after the first $j-1$ jobs were assigned; assign job $j$ to machine $i$ which minimizes the value $b_i \cdot W_i^{(j-1)} + b_i \cdot l_j$ (i.e., minimizing the time when machine $i$ finishes when job $j$ is assigned to it) where ties are broken arbitrarily.

Can you design prices such that this algorithm and the designed prices form a truthful mechanism?

**EXERCISE 11.4:** (6 points)
In the Generalized Second-Price (GSP) Auction, $n$ advertisement slots with click-through rates $r_1 \geq r_2 \geq \ldots \geq r_n \geq 0$, are auctioned off to $n$ buyers with per-click valuations $v_1 \geq v_2 \geq \ldots \geq v_n > 0$. In GSP, each buyer $i$ cast a bid $b_i$, the mechanism sorts the bids in a decreasing order, assigns the $k$-th highest bidder the corresponding slot $k$, and charges it the price-per-click equal to the bid $b_{\sigma(k)+1}$ (with the convention that $b_{n+1} := 0$). Let $\pi(j)$ denote the player that is assigned to slot $j$, $j = 1, 2, \ldots, n$. (Observe that $\pi$ is a permutation of $\{1, 2, \ldots, n\}$.) Then, the valuation of player $\pi(j)$ of the slot $j$ is $v_{\pi(j)} \cdot r_j$, and the payment of the player is $b_{\pi(j+1)} \cdot r_j$. Thus, the player’s payoff is $v_{\pi(j)} \cdot r_j - b_{\pi(j+1)} \cdot r_j$.

Consider the GSP auction as a strategic game, where the strategies of players are the bids $b_i$, and the payoffs are given by the result of the GSP auction. The social welfare of a strategic profile $b = (b_1, \ldots, b_n)$ is the total valuation of the slots assigned to players, i.e., $\text{SW}(b) := \sum_{j=1}^n v_{\pi(j)} \cdot r_j$. An assignment of slots to bidders that maximizes this sum is called a social optimum, and denoted by OPT. Observe that an assignment where $\pi(j) = j$ is a social optimum, having social welfare $\sum_{j=1}^n v_j \cdot r_j$.

We are interested in price of anarchy (PoA), which is now defined as the ratio

$$\frac{\text{SW}(\text{OPT})}{\text{SW}(\text{worst NE})} = \frac{\sum_{j=1}^n v_j \cdot r_j}{\sum_{j=1}^n v_{\pi^*(j)} \cdot r_j},$$

where $\pi^*(j), j = 1, \ldots, n$ is the assignment of buyers to slots in a Nash equilibrium of largest social welfare.

a) Show that PoA can be arbitrary bad, i.e., show that for any $\alpha > 1$ there exists a setting in which the PoA is larger than $\alpha$.

(Hint: You may consider a Vickrey auction translated into GSP.)
b) Consider the restriction of GSP game in which every player $i$ can only bid $b_i \leq v_i$ (i.e., the set of strategies $S_i$ is equal to $\{x : x \leq v_i\}$.

i) Consider a Nash equilibrium, and let $\pi(j)$ be the buyer assigned to slot $j$ in this Nash equilibrium. Prove that for every $j$ and $j'$, the following holds:

$$v_{\pi(j')} \cdot r_{j'} + v_{\pi(j)} \cdot r_j \geq v_{\pi(j')} \cdot r_j,$$

or, equivalently,

$$\frac{r_{j'}}{r_j} + \frac{v_{\pi(j)}}{v_{\pi(j')}} \geq 1.$$

(Hint: Use the fact that in NE no player wants to change its strategy to obtain a different slot.)

ii) Using the above inequality, prove that in any restricted GSP game, the price of anarchy is at most two.