EXERCISE 10.1:
In this exercise we consider combinatorial auctions with single-minded bidders. Recall that in such an auction every player is only interested in getting the goods in \( S_i \subseteq U \) (where \( U \) is the set of goods). The player \( i \) values this bundle \( S_i \) with \( v_i \in \mathbb{R}^+ \). Both \( S_i \) and \( v_i \) are the private information of player \( i \). Every player \( i \) submits a bid \((B_i, b_i)\) to the auction, expressing the desire to get the bundle \( B_i \) and that the player values it with \( b_i \).

Recall the characteristics of VCG and the LOS mechanisms. In VCG the mechanism computes an optimal allocation \( \{S^*_i\}_{i=1}^n \) of goods to the players (where the allocation maximizes the sum of the valuations of all players), and the payment \( p_i \) to every player \( i \):

\[
p_i = \sum_{j \neq i} b_j(S_j) - \sum_{j \neq i} b_j(S^*_j),
\]

where \( \{S_j\}_{j \neq i} \) is an assignment maximizing the total valuation of players 1, 2, \ldots, \( i-1, i+1, \ldots, n \).

In a LOS mechanism a greedy algorithm is used to compute an approximate solution. In each iteration it grants the bid with the highest value according to the formula \( b_i/\sqrt{|B_i|} \), after which it removes the bids that are blocked by \( B_i \) before reiterating. The payment to a player \( i \) is then \( q_i = b_j\sqrt{|B_i|/|B_j|} \), where player \( j \) is the highest uniquely blocked bidder of \( i \). In both mechanisms a player who is not granted his bid pays nothing.

a) Consider the VCG mechanism and the LOS mechanism for a combinatorial auction with single-minded bidders.

Provide a problem instance for each one of the following settings:

i) The total sum of payments in the VCG mechanism is greater than the total sum of payments in the LOS mechanism.

ii) The total sum of payments in the LOS mechanism is greater than the total sum of payments in the VCG mechanism.

b) Consider the LOS greedy algorithm for granting bids of players. In the lecture we have seen that a player \( i \) with her bid \((B_i, b_i)\) can uniquely block (u-block for short) a player \( j \) with her bid \((B_j, b_j)\) even if \( B_i \cap B_j = \emptyset \). Show, however, that if \( j \) is the highest u-blocked bid by player \( i \), then \( B_i \cap B_j \neq \emptyset \).