Exercise 11.1  Max-Flow by Hand.

Apply one of the algorithms presented in the lecture for finding a maximum flow from $s$ and $t$ in the following network. The capacities are given next to the corresponding edges. Provide the resulting maximum flow, minimum cut and residual graph.

Exercise 11.2  Championship Problem.

Until 1995, in the swiss soccer national league A the so-called 2 point rule was used: A win gives 2 points, a tie gives 1 point and a loss gives 0 points. If two teams have the same number of points, then the better position in the table is assigned to the team with the better goal difference, i.e. the difference of scored goals minus the number of conceded goals. Suppose that there were two more games and the current table looked like this:

<table>
<thead>
<tr>
<th>Team</th>
<th>Points</th>
<th>Next opponents</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) FC St. Gallen (FCSG)</td>
<td>37</td>
<td>FCB, FCW</td>
</tr>
<tr>
<td>2) BSC Young Boys (YB)</td>
<td>36</td>
<td>FCW, FCB</td>
</tr>
<tr>
<td>3) FC Basel (FCB)</td>
<td>35</td>
<td>FCSG, YB</td>
</tr>
<tr>
<td>4) FC Luzern (FCL)</td>
<td>33</td>
<td>FCZ, GCZ</td>
</tr>
<tr>
<td>5) FC Winterthur (FCW)</td>
<td>31</td>
<td>YB, FCSG</td>
</tr>
</tbody>
</table>

At first glance, FC Luzern may win the championship if both next games are won and FC St. Gallen loses both next games (in this case, both had 37 points, but FC Luzern had the better goal difference).

Model the situation above as a flow problem and specify the value that a maximum flow needs to have such that FC Luzern wins the championship. Use that in every game exactly two points are distributed among the opponents. Use the max-flow min-cut theorem to conclude that FC Luzern cannot win the championship in any case.

Note: A comparable real situation existed in the german soccer national league 1964/65.
Exercise 11.3  Assigning Students to Courses.

Consider a university with $n$ students. Every student has to attend exactly 5 courses in the next semester. In overall there exist $m$ courses, and the course $k$ can be attended by at most $T_k \in \mathbb{N}$ students.

Every student $i$ is interested in a set $K_i$ of 10 courses. The university wants to assign courses to the students such that every student attends only courses he likes, and that no course $k$ is attended by more than $T_k$ students.

a) First, we want to decide whether such an assignment of courses to students exists. Model this problem as a flow problem. Explain carefully how a network $N = (V, E)$ can be constructed from $K_1, ..., K_n$ and $T_1, ..., T_m$. Specify the vertices $V$ and the edges $E$ of the construction and describe which capacities have to be assigned to the edges. Describe how you can conclude from the value of a maximal flow whether such an assignment exists or not.

b) Specify a flow algorithm that solves (a) as efficient as possible. Assume that there are more students than courses, and specify the running time of the above-mentioned algorithm in dependency of $n$ and $m$.

c) Assuming that an appropriate assignment of students to courses exists, we are interested in computing this assignment. Describe an algorithm that computes for every student $i$ the set of the 5 courses that $i$ has to attend in the next semester. Which running time does your method have, if a maximal flow in the network $N$ has been already computed before?

Exercise 11.4  Matchings.

a) Provide a connected graph with 6 vertices that contains exactly 3 different perfect matchings.

b) Provide a subset of vertices of the following bipartite graph that using Hall’s theorem prove that the graph has no perfect matching.

Hand-in: until Wednesday, 22nd May 2013.