There is a definition of the $\mathcal{O}$ notation that is different from the one given at the lecture. Namely, for a function $g : \mathbb{N} \rightarrow \mathbb{R}^+$, let

$$\mathcal{O}(g) := \{ f : \mathbb{N} \rightarrow \mathbb{R}^+ \mid \exists c \in \mathbb{R}^+, n_0 \in \mathbb{N} \forall n \geq n_0 : f(n) \leq cg(n) \}.$$  

Analogously, we say that a function $f$ grows asymptotically at least as much as $g$, if $f \in \Omega(g)$ with

$$\Omega(g) := \{ f : \mathbb{N} \rightarrow \mathbb{R}^+ \mid \exists c \in \mathbb{R}^+, n_0 \in \mathbb{N} \forall n \geq n_0 : f(n) \geq cg(n) \}.$$  

A function $f$ grows asymptotically like $g$ when $f \in \mathcal{O}(g)$ and $f \in \Omega(g)$. We will write this as $f \in \Theta(g)$, or as $f = \Theta(g)$.

For these exercises, you can choose to use the definition given at the lecture, or use the above definition.

**Exercise 1.1**  The Set $\Theta(g)$.

Give a definition of the set $\Theta(g)$ as compactly as possible (i.e., with the fewest possible parameters and quantifiers), analogously to the above definitions for the sets $\mathcal{O}(g)$ and $\Omega(g)$.

**Exercise 1.2**  Proofs about $\mathcal{O}$ Notation.

Prove or disprove the following statements, where $f, g : \mathbb{N} \rightarrow \mathbb{R}^+$.

- a) $f \in \mathcal{O}(g)$ if and only if $g \in \Omega(f)$.
- b) If $f \leq g$, then $f \in \mathcal{O}(g)$.
- c) If $f \in \mathcal{O}(g)$, then $f \leq g$.
- d) There exist different functions $f$ and $g$ such that $f \in \Omega(g)$ and $g \in \Omega(f)$.
- e) For every $a, b \in \mathbb{N}$, $a \leq b$ it holds that $\sqrt[n]{a} \in \Theta(\sqrt[n]{b})$.
- f) For every $a, b \in \mathbb{N}$ it holds that $\log_a(n) \in \Theta(\log_b(n))$.
- g) Let $p(n) := \sum_{k=0}^{d} a_k n^k$, $a_0, ..., a_d \in \mathbb{R}$ with $a_d > 0$ be a polynomial of degree $d$. Then $p \in \Theta(n^d)$.
- h) $k^n \in \mathcal{O}(2^n)$ for constant $k \in \mathbb{N}$, $k > 2$.

**Exercise 1.3**  Asymptotic growth of functions.

Sort the following functions from left to right such that: if function $f$ is on the left of $g$, then $f \in \mathcal{O}(g)$.

*Example:* the functions $n^3$, $n^7$, $n^9$ are already in the right order since $n^3 \in \mathcal{O}(n^7)$ and $n^7 \in \mathcal{O}(n^9)$.

- $n^{0.1}$, $5^n$, $n!$, $\log(n^n)$, $n^n$, $\left(\frac{n}{5}\right)$, $\log(n^{15})$, $\log^2(n)$, $\sqrt[n]{n}$

**Exercise 1.4**  $o$ Notation.

Apart from the already presented $\mathcal{O}$ notation there also exists $o$ notation. For two functions $f, g : \mathbb{N} \rightarrow \mathbb{R}^+$, we have $f \in o(g)$ if and only if $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$. Informally spoken, $f$ grows slower than $g$. Analogically with $\Omega$ notation, we define $f \in \omega(g)$ if and only if $\lim_{n \to \infty} \frac{g(n)}{f(n)} = 0$. Prove the following statements.
a) $f \in o(g) \Rightarrow f \in \mathcal{O}(g)$, but $f \in \mathcal{O}(g) \nRightarrow f \in o(g)$.

b) Let $p(n) := \sum_{k=0}^{d} a_k n^k$, $a_0, \ldots, a_d \in \mathbb{R}$ be a polynomial of degree $d$. Then $p \in o(n^{d+1})$.

c) Let $k > 0, b > 1$ be arbitrary constants. Then $n^k \in o(b^n)$.

d) $\log(n) \in o(n)$

Note: To prove the statements (c) and (d), the ratio criterion for sequences may be helpful: let $(a_n)_{n \geq 1}$ be a real-valued sequence. If there exist a constant $c < 1$ and an index $n_0 \in \mathbb{N}$ such that $\left| \frac{a_{n+1}}{a_n} \right| \leq c$ for every $n \geq n_0$, then $(a_n)$ converges to zero.