There is a definition of the $O$ notation that is different from the one given at the lecture. Namely, for a function $g : \mathbb{N} \to \mathbb{R}^+$, let
\[ O(g) := \{ f : \mathbb{N} \to \mathbb{R}^+ | \exists c \in \mathbb{R}^+, n_0 \in \mathbb{N} \forall n \geq n_0 : f(n) \leq cg(n) \} . \] (1)

Analogously, we say that a function $f$ grows asymptotically at least as much as $g$, if $f \in \Omega(g)$ with
\[ \Omega(g) := \{ f : \mathbb{N} \to \mathbb{R}^+ | \exists c \in \mathbb{R}^+, n_0 \in \mathbb{N} \forall n \geq n_0 : f(n) \geq cg(n) \} . \] (2)

A function $f$ grows asymptotically like $g$ when $f \in O(g)$ and $f \in \Omega(g)$. We will write this as $f \in \Theta(g)$, or as $f = \Theta(g)$.

Note:
1) For these exercises, you can choose to use the definition given at the lecture, or use the above definition.
2) l’Hôpital’s rule may be helpful: Let $f, g : \mathbb{R} \to \mathbb{R}$ be two differentiable functions with $f(x) \to \infty$ and $g(x) \to \infty$ for $x \to \infty$. If $\lim_{x \to \infty} \frac{f'(x)}{g'(x)}$ exists, then we have
\[ \lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)} . \] (3)

Exercise 1.1  The Set $\Theta(g)$.

Give a definition of the set $\Theta(g)$ as compactly as possible (i.e., with the fewest possible parameters and quantifiers), analogously to the above definitions for the sets $O(g)$ and $\Omega(g)$.

Exercise 1.2  Proofs about $O$ Notation.

Prove or disprove the following statements, where $f, g : \mathbb{N} \to \mathbb{R}^+$.

a) $f \in O(g)$ if and only if $g \in \Omega(f)$.

b) If $f(n) \leq g(n)$ for every $n \in \mathbb{N}$, then $f \in O(g)$.

c) If $f \in O(g)$, then $f(n) \leq g(n)$ for every $n \in \mathbb{N}$.

d) There exist different functions $f$ and $g$ such that $f \in \Omega(g)$ and $g \in \Omega(f)$.

e) $\log_a(n) \in \Theta(\log_b(n))$ for all constants $a, b \in \mathbb{N}$.

f) Let $f_1, f_2 \in O(g)$ and $f(n) := f_1(n) + f_2(n)$. Then, $f \in O(g)$.

g) Let $f_1, f_2 \in O(g)$ and $f(n) := f_1(n) \cdot f_2(n)$. Then, $f \in O(g)$.

h) $k^n \in O(2^n)$ for constant $k \in \mathbb{N}$, $k > 2$.

Please turn over.
Exercise 1.3  Asymptotic Growth of Functions I.

Prove the following statements:

a) Let \( f, g : \mathbb{N} \rightarrow \mathbb{R}^+ \) be two functions with \( \lim_{n \to \infty} g(n) \neq 0 \). If the sequence \( \frac{f(n)}{g(n)} \) converges to a constant \( C \geq 0 \), then \( f \in \mathcal{O}(g) \).

b) For all constants \( k \geq 0 \) and \( d > 0 \), we have \( \log^k(n) \in \mathcal{O}(n^d) \).

c) For all constants \( d \geq 0 \) and \( b > 1 \), we have \( n^d \in \mathcal{O}(b^n) \).

Exercise 1.4  Asymptotic Growth of Functions II.

Sort the following functions from left to right such that: if function \( f \) is on the left of \( g \), then \( f \in \mathcal{O}(g) \).

Example: the functions \( n^3, n^7, n^9 \) are already in the right order since \( n^3 \in \mathcal{O}(n^7) \) and \( n^7 \in \mathcal{O}(n^9) \).

\[
n^3 + n, \frac{3^n}{n^3}, \sqrt{3n}, n!, \log(n^n), n^n, 10^{10}, \left( \frac{n}{5} \right), \log(n^{15}), \log^7(n), \frac{\sqrt{n}}{\log^3(n)}\]

Hand-in: until Wednesday, 26th February 2014.