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Deadline: Beginning of next lecture

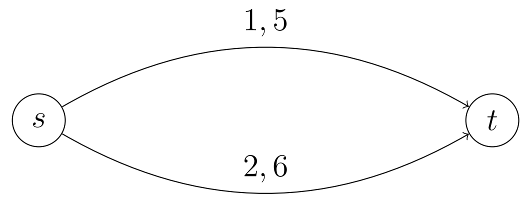
Algorithmic Game Theory

Fall 2015

Exercise Set 4

Exercise 1: (1+2+1 Points)

Consider this symmetric network congestion game with two players:



(a) What are the price of anarchy and the price of stability for pure Nash equilibria?

(b) What are the price of anarchy and the price of stability for mixed Nash equilibria?
Hint: Start by listing all mixed Nash equilibria. To obtain these start with a sentence like, "*Let σ be a mixed Nash equilibrium with $\sigma_1 = (\lambda_1, 1-\lambda_1)$, $\sigma_2 = (\lambda_2, 1-\lambda_2)$*," and continue by deriving properties of λ_1 and λ_2.

(c) What is the best price-of-anarchy bound that can be shown via smoothness?

Exercise 2: (3 Points)

For every $M \geq 1$, give an example of a two-player network congestion game whose price of anarchy for pure Nash equilibria is at least M.

Exercise 3: (3 Points)

Fair cost sharing games are a congestion games with delay functions are of the form

$$d_r(x) = c_r/x$$

where c_r is a positive constant. (In these games, c_r represents the cost for building resource r, and this cost is shared equally among the players using this resource.)

(a) Show that fair cost sharing games with n players are $(n, 0)$-smooth.

(b) For every n, give an example of an n-player fair cost sharing game whose price of anarchy for pure Nash equilibria (PoA_{PNE}) is at least n.

Exercise 4: (3 Points)

An ϵ-Nash equilibrium is a strategy profile s such that

$$c_i(s)(1-\epsilon) \leq c_i(s'_i, s_{-i})$$

for all players i and for all $s'_i \in S_i$. Prove that in congestion games with affine delay functions the cost of any ϵ-Nash equilibrium is at most $\frac{5}{2-3\epsilon}$ times the optimal social cost. (The social cost is the sum of all players' costs.)

Exercise 5: (2 Points)

Consider the variant of atomic congestion games in which the cost of a strategy profile is not the sum of all players' costs, but the *maximum*:

$$cost(s) := \max_{i \in \mathcal{N}} c_i(s). \tag{1}$$

Exhibit a class of congestion games for which the price of anarchy for coarse correlated equilibria and this notion of social cost is higher than the price of anarchy defined with respect to the sum of the players' costs.

Exercise 6: (4 Points)

In this exercise we consider *load balancing games* (already seen in Exercise Sheet 2): There are m machines of identical speeds. Player i is in charge of one job of weight $w_i > 0$. Every player may choose a machine to process this job; his strategy set is therefore $\{1, \ldots, m\}$. Player i's cost in strategy profile s is given as

$$c_i(s) = load_{s_i}(s) := \sum_{i' : s_{i'} = s_i} w_{i'} \ .$$

(Note that this is the load of the machine chosen by i, including w_i.)

Here we define the cost of a strategy profile as the *maximum load* among all machines:

$$cost(s) = \max_{\ell \in \{1, \ldots, m\}} load_\ell(s) \ . \tag{2}$$

Show that in these games $PoA_{\mathsf{PNE}} \leq 2$ for any number m of machines, where PoA is the price of anarchy defined with respect to the cost function in (2).