Paul Dütting, ETH Zürich October 19, 2015
Deadline: Beginning of next lecture

Algorithmic Game Theory

Fall 2015

Exercise Set 5

Exercise 1: (1+3 Points)
Consider the following instance of a *loadbalancing game*. There are four players $\mathcal{N} = \{1, 2, 3, 4\}$. Players $i \in \{1, 2\}$ have a job with weight $w_i = 2$, and players $i \in \{3, 4\}$ have a job with weight $w_i = 1$. Every player chooses a machine to process his job on; his strategy set is therefore $\{1, 2\}$. For a given strategy profile s the cost of player i is $c_i(s) = load_{s_i}(s) = \sum_{i':s_{i'}=s_i} w_{i'}$ and the overall goal is to minimize the makespan $cost(s) = \max\{load_1(s), load_2(s)\}$.

(a) Show that the price of anarchy for pure Nash equilibria is 4/3.

(b) Show that the price of anarchy for mixed Nash equilibria is strictly larger than 4/3.

Exercise 2: (5+2 Points)
Now consider the general case of a *loadbalancing game* with n players $\mathcal{N} = \{1, 2, \ldots, n\}$ and m machines, in which each player $i \in \mathcal{N}$ has a job with weight w_i and has to choose a machine $j \in [m]$. The cost of player i under a given strategy profile is $c_i(s) = load_{s_i}(s) = \sum_{i':s_{i'}=s_i} w_{i'}$ and the goal is to minimize the makespan $cost(s) = \max_l load_l(s)$. Consider the Largest Processing Time (LPT) scheduling algorithm. This algorithm inserts the jobs in a non-increasing order of weights, assigning each job to a machine that minimizes the cost of the job at its insertion time.

(a) Show that this algorithm computes a pure Nash equilibrium.

(b) Use (a) to bound the price of stability for pure Nash equilibria.

Exercise 3: (2+6 Points)
We now extend our analysis of *loadbalancing games* to the case where the weight of a job depends on the machine it is processed on. Formally, we are given a set of players $\mathcal{N} = \{1, \ldots, n\}$ and m machines. Each player i has a job, the weight of this job on machine j is $w_{i,j} > 0$. As before the strategy set of each player is $\{1, \ldots, m\}$. For a given strategy profile the cost of player i is $c_i(s) = load_{s_i}(s) = \sum_{i':s_{i'}=s_i} w_{i',s_{i'}}$ and our objective is to minimize the makespan $cost(s) = \max_l load_l(s)$.

(a) Show that the price of anarchy for pure Nash equilibria is unbounded, even if there are only $m = 2$ machines.

(b) Show that for $m = 2$ machines the price of anarchy for strong Nash equilibria is at most 2.

Exercise 4: (1 Points)
Consider a single-item, first-price auction with n bidders. Show by means of an example that truthful bidding by all players need not constitute a pure Nash equilibrium.

Exercise 5: (3 Points)
Consider a single-item, first-price auction with n bidders. For $\epsilon > 0$ a pure ϵ-Nash equilibrium is a profile of bids b such that for all bidders i and all possible deviations b_i' it holds that $u_i(b, v_i) \geq u_i((b_i', b_{-i}), v_i) - \epsilon$. Show that independent of the tie-breaking rule, for any $\epsilon > 0$, there exists a pure ϵ-Nash equilibrium in which the bidder i with the highest valuation $v_i \geq \max_j v_j$ wins the item and his payment p_i^{1st} is $p_i^{1st} \leq p_i^{2nd} + \epsilon$ where p_i^{2nd} is his payment in the truthful equilibrium of the second-price auction.