Paul Dütting, ETH Zürich November 9, 2015

Algorithmic Game Theory

Fall 2015

Graded Problem Set

Your solutions to this exercise sheet will be graded. This grade will account for 15% of your final grade for the course. You are expected to solve them carefully and then write a nice complete exposition of your solution using **LaTeX**. The appearance of your solution will also be part of the grade. You are welcome to discuss the tasks with your fellow students, but we expect each of you to hand in your own individual write-up. Your write-up should list all collaborators. The deadline for handing in the solution is **Wednesday Nov 18, before midnight**. Please send your PDF via email to `akaki@inf.ethz.ch`. Your file should have the name <Surname>.pdf, where <Surname> should be exchanged with your family name.

Problem 1: (8 Points)

In the following 2-player normal form game. A,B,C,D are the payoffs to player I, which are real numbers, no two of which are equal. Similarly, a,b,c,d are the payoffs to player II, which are real numbers, also no two of which are equal.

I \ II	left	right
Top	A, a	B, b
Bottom	C, c	D, d

1. Under which conditions does this game have a mixed Nash equilibrium which is not a pure Nash equilibrium? (3 Pt)
2. Under which conditions in 1. is this the only Nash equilibrium of the game? (3 Pt)
3. Consider one of the situations in 2. and compute the probabilities $1-p$ and p for playing "Top" and "Bottom", respectively, and $1-q$ and q for playing "left" and "right", respectively, that hold in equilibrium. (2 Pt)

Hint: When trying to answer questions 1. and 2. you may find it helpful to draw arrows alongside the sides of the game to indicate a player's preference given a pure strategy chosen by the other player.

Problem 2: (12 Points)

In this question we consider *network design games* with k players on *undirected* graphs $G = (V, E)$. Every edge $e \in E$ has a positive cost $c_e \in \mathbb{R}$. Every player i has a pair of vertices $s_i, t_i \in V$, which she wants to connect by a path in G: a strategy of player i is a path P_i from s_i to t_i in G. Every player pays his fair share for every edge of the chosen path, i.e., if an edge $e \in E$ of cost c_e is chosen by k_e players (in their chosen paths) in a strategy profile $s = (P_1, \ldots, P_N)$, then every such player pays c_e/k_e as a contribution towards the total cost of the edge. The total cost $c_i(s)$ of player i in strategy profile s is then the sum of all her contributions. The social cost $SC(s)$ of strategy profile s is $\sum_i c_i(s)$.

1. Prove that if G is a tree then the *price of stability* is 1. (1 Pt)
2. From now on assume that G is a cycle, all target vertices t_i are the same, all s_i are pairwise different and edge costs are positive.
   (a) Provide an example of such a game in which the price of stability is greater than 1. (1 Pt)
   (b) Prove that the price of stability is at most 2 in this case. (4 Pt)
   (c) Design a linear time algorithm in $|V|$ for finding an optimum social-cost strategy profile. (2 Pt)
   (d) Show that no Nash equilibrium strategy-profile contains all the edges of G. (2 Pt)
   (e) Design a polynomial time algorithm in $|V|$ for calculating the price of stability of any instance of such a game. (2 Pt)

Problem 3: (10 Points)
Consider an auction of one mp3 song. The song can be copied and sold to as many bidders as desired. Thus, the setting can be seen as an auction of n players (bidders) with n identical items, where every player wants exactly one item. Each player i has a private valuation $v_i \in \mathbb{R}^+$ for getting the song. The seller incurs a cost 0, if no one gets the song (i.e., if it does not sell a single copy of the song), and it incurs a cost 1, if one or more bidders get the song. We define the *surplus* as a function of the winners W of the auction as follows:

$$\text{surplus}(W) = \begin{cases} 0 & \text{if } W = \emptyset, \\ -1 + \sum_{i \in W} v_i & \text{else.} \end{cases}$$

1. Design a truthful mechanism that maximizes the surplus and for which the utility of every bidder is non-negative. (2 Pt)
2. A mechanism is said to have a balanced budget if the payments collected by the winners cover the costs of the seller. Is your mechanism from 1. budget-balanced? (2 Pt)
3. Show that the following mechanism is truthful and budget-balanced: (2 Pt)
   - Collect all bids $\mathbf{b} = (b_1, \ldots, b_n)$.
   - Set W to be all bidders.
   - Iterate (While $W \neq \emptyset$):
     (a) Set $p = 1/|W|$;
     (b) If $b_i \geq p$ for every $i \in W$, output W as winners and charge them p;
     (c) Else remove arbitrary i with $b_i < p$ from W
   - If $W = \emptyset$, then there are no winners (and no payments).
4. The mechanism proposed in 3. comes relatively close to maximizing the surplus. Show that, in the worst-case, the difference between the maximum surplus and the surplus of the mechanism in 3. is at most $-1 + \sum_{i=1}^{n} \frac{1}{i} = H_n - 1$. (4 Pt)