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Deadline: Beginning of next lecture

Algorithmic Game Theory

Fall 2015

Exercise Set 12

Exercise 1: (5 Points)

Consider the *House Allocation* problem with n players and the following algorithm that is equivalent to the *Top Trading Cycle algorithm* (TTCA) presented in the lecture. The algorithm operates on the following (complete) directed graph, where every player i (and its house) is represented by a vertex i. If house j is player i's kth choice, we add a directed edge (i, j) of color k. The algorithm works as follows: in every iteration $i = 1, \ldots, n$ every player considers her best option (i.e., the outgoing edge of smallest color) in the current graph. The considererd edges induce node-disjoint directed cycles and loops. Let N_i be the set of players that form these cycles in iteration i. The algorithm reassigns the houses to the players in N_i consistently according to their preferences (according to the selected edges). Before starting the next iteration, the algorithm removes the nodes corresponding to N_i (and their incident edges) from the graph and it increases i.

1. Apply the TTCA to the following instance with players a, b, c, d.

$$\begin{aligned} a &: \; b \succ c \succ a \succ d \\ b &: \; c \succ a \succ b \succ d \\ c &: \; d \succ a \succ c \succ b \\ d &: \; d \succ c \succ a \succ b \end{aligned}$$

2. Prove that the outcome of the TTCA is in the core of the House Allocation problem, that is, there is no blocking coalition S among the players for the allocation produced by TTCA.

3. Consider the following modified version of the TTCA (which is the algorithm given in the AGT book). At every iteration i we look only at the loops and cycles that have color i and again we set N_i as the set of players in that loops and cycles. We reallocate the houses among N_i in the natural way, following the preferences induced by the directed edge in every such cycle (loop). After the corresponding reallocations have been done, we remove all the edges of color i and all players N_i and we iterate.

   Does the output of this algorithm belong to the core of the House Allocation problem? Provide an argument or a counterexample.

Exercise 2: (6 Points)

Consider the following modification of the *house allocation problem*. There are n players $N = \{1, 2, \ldots, n\}$, each owning a house. Moreover, there is a hotel offering each player i a room with full service in exchange for the house of player i (The hotel is not a player in this setting, however). Each player i has a strict preference over all the houses and a hotel room with full service. A mechanism for this setting asks the players for these preferences, and as a result it produces an allocation of each player to a house or to a hotel room. Here, each house can be assigned to at most one player, but hotel rooms can be assigned to more than one player (there are enough rooms in the hotel for all players). Each player is interested to get its most preferred option.

Consider the following modification of the top trading-cycle algorithm (TTCA). The algorithm works in rounds $r = 1, 2, \ldots$. In every round, some of the players get allocated to a house or to a hotel room and will not be considered by the algorithm anymore. In round r, we construct the *most-preferred graph*, i.e., a graph where every player that has not been allocated yet is present as a vertex, and additionally, there is a vertex representing the hotel. For each player i we consider the most preferred allocation j among the remaining options and add a directed edge (i, j) to the graph. Consider all cycles and self-loops in the most-preferred graph. These cycles and loops suggest an allocation: assign each player i from a cycle (loop) the house or hotel room j to which it points with the directed edge (i, j). Remove all players that were allocated from the consideration of the algorithm, and proceed with round $r + 1$ until there are no players left.

1. Show that in this setting it can happen that there are no cycles or self-loops in the most-preferred graph. Show that if this is the case, then the most-preferred graph contains trees where every directed path of the tree ends in the hotel.

2. Consider the following modification of TTCA. If in round r there are cycles or loops, proceed as above. If, however, there are no cycles and loops, do the following: choose a path ending in the hotel room; for each directed edge (i, j) of the path, allocate the option j to player i (i.e., either the house of j goes to i if j is a player, or the hotel room goes to i if j is the hotel); proceed with another such path (if there is any); the house of the last player in this path (last when ordered from the hotel) goes to the hotel. The question is which path shall the algorithm choose. We consider the following two variants:

   (a) Consider the rule where we always select a longest path, i.e., a path which contains a maximum number of edges among all existing paths. In case of ties, the path whose first player has the smallest index is chosen. Show that this algorithm does not give a truthful mechanism.

   (b) Consider the rule where we always select an edge leading to the hotel room as the path. (Thus, the path does not have to start in a leaf of the tree!) Show that this algorithm gives a truthful mechanism.
   (You may use the fact that the original TTCA induces a truthful mechanism.)

Exercise 3: (6 Points)

a) It was shown in the lecture that the *deferred-acceptance* algorithm when men propose is not truthful for women, and an example has 3 men and 3 women. Is it truthful for 2 men and 2 women?

b) Construct an example in which there are 2 different stable matchings.