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Deadline: Beginning of next lecture

Algorithmic Game Theory

Fall 2016

Exercise Set 2

Exercise 1: (3 Points)
For every $M \geq 1$, give an example of a two-player network congestion game whose price of anarchy for pure Nash equilibria is at least M.

Exercise 2: (3 Points)
Fair cost sharing games are congestion games with delay functions of the form

$$d_r(x) = c_r/x$$

where c_r is a positive constant. (In these games, c_r represents the cost for building resource r, and this cost is shared equally among the players using this resource.)

(a) Show that fair cost sharing games with n players are $(n, 0)$-smooth.

(b) For every n, give an example of a fair cost sharing game with n players whose price of anarchy for pure Nash equilibria (PoA) is at least n.

Exercise 3: (3 Points)
An ϵ-Nash equilibrium is a state $s \in S$ such that

$$c_i(s)(1-\epsilon) \leq c_i(s'_i, s_{-i})$$

for all players i and for all $s'_i \in S_i$. Prove that in congestion games with affine delay functions the cost of any ϵ-Nash equilibrium is at most $\frac{5}{2-3\epsilon}$ times the optimal social cost. (The social cost is the sum of all players' costs.)

Exercise 4: (3 Points)
In this exercise we consider *load balancing games*: There are m machines of identical speeds. Player i is in charge of one job of weight $w_i > 0$. Every player may choose a machine to process this job; his strategy set is therefore $\{1, \ldots, m\}$. Player i's cost in state s is given as

$$c_i(s) = load_{s_i}(s) := \sum_{i' : s_{i'} = s_i} w_{i'} \ .$$

(Note that this is the load of the machine chosen by i, including w_i.)
Prove that every pure Nash equilibrium is a local minimum for the function f given by the sum of the *squares of the machine loads*:

$$f(s) = \sum_{\ell=1}^{m} \left(load_\ell(s)\right)^2 .$$

Explain how this implies that a pure Nash equilibrium exists. Does f satisfy the definition of potential game given in the lecture (Lecture 2)?

Exercise 5: (3 Points)
Consider again load balancing games introduced in the previous exercise. Now define the cost of a strategy profile as the *maximum load* among all machines:

$$cost(s) = \max_{\ell \in \{1,\ldots,m\}} load_\ell(s) \ . \tag{1}$$

Show that in these games $PoA \leq 2$ for any number m of machines, where PoA is the price of anarchy for pure Nash equilibria defined **with respect to the cost function in** (1).

Exercise 6: (4 Points)
Prove that computing pure Nash equilibria in congestion games remains PLS-complete also when we restrict to **affine delay functions**. That is, for $d_r(x) = a_r x + b_r$ with $a_r, b_r \geq 0$.