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Deadline: Beginning of next lecture

Algorithmic Game Theory

Fall 2016

Exercise Set 3

Exercise 1: (3 + 1 Points)

Consider the following cost-minimization game between two players, "row" and "column." Both players have four strategies $S_{row} = S_{col} = \{A, B, C, D\}$. For each strategy profile $(s_{row}, s_{col}) \in S \times S$ the entry in the lower left corner of the corresponding cell in the table below denotes the **cost** to the row player and the entry in the upper right corner the **cost** to the column player.

	A	B	C	D
A	6 4	5 6	4 5	5 6
B	6 5	5 1	1 2	5 5
C	6 6	3 5	3 3	2 1
D	6 7	3 7	5 6	7 7

a) Find all pure Nash equilibria of this game. Explain why exactly those states are pure Nash equilibria and why no other state is one.

b) Find all mixed Nash equilibria of this game. Explain why these probability distributions are the only mixed Nash equilibria.

Exercise 2: (2 + 2 + 2 Points)

Consider the following cost-minimization game, played between two drivers approaching a crossing. The drivers can either stop (S) or cross immediately (C). If they both cross immediately, then they build an accident. This has, of course, high cost for both. If one crosses and one stops, then the player which stops has a small cost for waiting. If both stop, then both have to wait and suffer a small cost.

	C(ross)	S(top)
C(ross)	100 100	1 0
S(top)	0 1	1 1

The numbers in the table are **costs** for the players.

(a) List all pure and mixed Nash equilibria.

(b) Give a correlated equilibrium that is not a mixed Nash equilibrium.

(c) Argue that in this game the set of coarse correlated equilibria coincides with the set of correlated equilibria.

Exercise 3: (4 Points)

Consider a symmetric network congestion game with four players. Suppose the network consists of the source s, the target t, and six parallel edges from s to t each with cost function $c(x) = x$. Consider the distribution σ over states that randomizes uniformly over all states with the following properties:

- There is one edge with two players.
- There are two edges with one player each (so three edges are empty).
- The set of edges with at least one player is either $\{1, 3, 5\}$ or $\{2, 4, 6\}$.

Prove that σ is a coarse correlated equilibrium but not a correlated equilibrium.

Exercise 4: (2 Points)

Give the proof of the theorem on PoA_{CCE} for smooth games stated during the lecture:

> **Theorem.** *In a (λ, μ)-smooth game, the PoA for coarse correlated equilibria (PoA_{CCE}) is at most*
> $$\frac{\lambda}{1-\mu} .$$

Hint: Use linearity of expectation $\mathbf{E}[\cdot]$ and adapt the proof for PoA_{PNE}.

Exercise 5: (2 Points)

The multiplicative-weights algorithm presented in the lecture was stated such that the overall length of the sequence T is given as a fixed parameter. Give a no-external regret algorithm that works without such a parameter for all possible T.

Hint: Use the algorithm from class as a subroutine (you do not need to analyze it again). Start with $T = 1$ as a guess and run the subroutine. Once the subroutine ends, restart it but double your guess.

Exercise 6: (4 Points)

Consider the following algorithm for minimizing external regret:

Greedy Algorithm

- Set $p^1(a) = 1$ for $a = 1$, and $p^1(a) = 0$ for all $a \neq 1$.
- For each time step $t = 1, \ldots, T$:
- Let $C^t_{BEST} = \min_a c^t(a)$ and let $S^t = \{a \mid c^t(a) = C^t_{BEST}\}$. Set $p^{t+1}(a) = 1$ for $a = \min S^t$ and $p^{t+1}(a) = 0$ otherwise.

Show that the expected cost achieved by this algorithm is bounded by $m \cdot C^T_{BEST} + (m-1)$, where m is the number of possible actions (the possible actions are $\{1, 2, \ldots, m\}$).