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Algorithmic Game Theory

Fall 2016

Graded Problem Set

Your solutions to this exercise sheet will be graded. This grade will account for 15% of your final grade for the course. You are expected to solve them carefully and then write a nice complete exposition of your solution using **LaTeX**. The appearance of your solution will also be part of the grade. You are welcome to discuss the tasks with your fellow students, but we expect each of you to hand in your own individual write-up. Your write-up should list all collaborators. The deadline for handing in the solution is **Nov 17, before midnight**. Please send your PDF via email to `akaki@inf.ethz.ch`. Your file should have the name <Surname>.pdf, where <Surname> should be exchanged with your family name.

Problem 1: (3+1+1 Points)

Consider the following situation. Two radio stations want to transmit to a third station (receiver).

Each of the two transmitting stations can independently choose its own transmission power level among a set of values $S = \{0, 1, 2, \ldots, M\}$, where 0 means 'no transmission'. If both stations transmit simultaneously some interference occurs, and the receiver is only able to decode the message of the station that was using the **strongest power** among the two. Moreover, if both stations were using the **same power**, then **neither message** is received.

Each transmitter selfishly cares only about its own message and the power used to transmit (the power corresponds to the energy spent). That is, from the point of view of a transmitting station i, its cost is:

1. The power level s_i used to transmit;
2. An additional penalty $\kappa \in \mathbb{N}^+$ if the message of i is not received.

For example, if $s = (1, 3)$, then $c_1(s) = 1 + \kappa$ and $c_2(s) = 3$.

Your task is:

1. Prove that this game has a pure Nash equilibrium (PNE) if and only if $\kappa \in \{1, 2\}$.

   (Note that κ is always a positive integer and $M \geq 2$.)

2. For $\kappa = 1$ and $\kappa = 2$, find a tight bound on the price of anarchy for pure Nash equilibria.

   (Recall the definition from the lecture notes and that $cost(s) = \sum_i c_i(s)$ is the social cost.)

3. For $\kappa = 1$ and $M = 2$ give a **coarse correlated equilibrium** which has (1) optimal expected social cost and (2) the expected cost of the players is the same.

   (Prove that your equilibrium is a CCE and that it satisfies the requirements.)

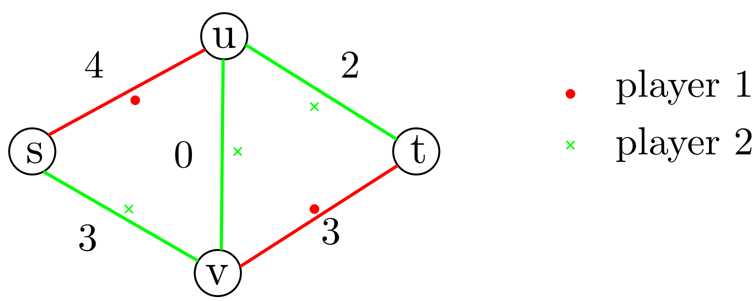
Problem 2: (3 Points)
Nash's Theorem states that every finite strategic game has a mixed Nash equilibrium. Prove this theorem for the special case in which we have only **two players** and each of them has only **two strategies**.

Problem 3: (2+2 Points)
Recall the definition of a **congestion game** (see Lecture 1). There is a set of n players, a set of m resources, where each resource r has delay function $d_r()$. For any state $s = (s_1, \ldots, s_n)$, let $n_r(s)$ denote the number of players using resource r. The cost of player i is equal to $c_i(s) = \sum_{r \in s_i} d_r(n_r(s))$, and the social cost of a state s is given by the sum of the costs of the players, i.e., $cost(s) = \sum_i c_i(s)$.

1. Show that for any constant integer $c > 0$ there is a congestion game with non-negative and increasing delays on each resource with c players such that Rosenthal's potential function has exactly c global minima.

2. Show that there is a congestion game in which there is a state s which is a global minimum of Rosenthal's potential function but not a minimum cost pure Nash equilibrium of the corresponding congestion game.

Problem 4: (4+2+3 Points)
Consider the following multi-parameter mechanism design problem. We are given a graph $G(V, E)$ and two nodes s and t, and each edge has a private cost which is known to the player owning this edge. We consider the case in which a player may own **several edges**. Here is an example with two players:



The feasible solutions consist of the paths connecting s to t, and the cost for a player is given by the sum of the costs of his/her edges in the chosen path. For example, the shortest path in the example above costs 0 to player 1 (red) and 5 to player 2 (green).

Here however we do not want to find the shortest path. We instead consider a different optimization goal, namely, we want to minimize the **maximum cost** among the players. In the example above, this would mean that we choose the lower path which results in a cost of 3 for both players (the maximum cost is 3 instead of 5).

Any solution a costs to player i

$$t_i(a) = \sum_{e \in E_i \cap a} t_i^e$$

where E_i is the subset of edges owned by i, a is a path (set of edges) connecting s to t, and t_i^e denotes the true cost of edge $e \in E_i$. We are interesting in minimizing the **maximum cost** among the players:

$$maxcost(a, t) := \max_i t_i(a)$$

Your task is:

1. Prove that no truthful mechanism (A, P) can minimize the maximum cost.
2. Give an n-approximate truthful mechanism, where n is the number of players.

   (An α-approximate mechanism guarantees, for every input t, $maxcost(A(t), t) \leq \alpha \cdot OPT(t)$, where $OPT(t) = \min_{a \in \mathcal{A}} maxcost(a, t)$ is the optimum maxcost.)
3. Strengthen your first result to show that for 2 players a truthful 2-approximation is the best possible.

   (You can solve this question directly and get also the points for the first one.)

Note: In this problem, each player i reports some cost c_i^e for each of his/her edges $e \in E_i$. Based on the reported costs c the mechanism selects a solution $A(c)$ and payments $P_i(c)$.