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Deadline: Beginning of next lecture

Algorithmic Game Theory

Fall 2016

Exercise Set 9

This exercise is on Lecture 8

Exercise 1: (5 Points)
Consider the **House Allocation** problem with n players and the following algorithm that is equivalent to the **Top Trading Cycle algorithm (TTCA)** presented in the lecture. The algorithm operates on the following (complete) directed graph:

- Every player i (and his/her house) is represented by a vertex i;
- If house j is player i's k^{th} choice, we add a directed edge (i, j) of color k.

The algorithm works as follows:

TTCA:

1. In every iteration $i = 1, \ldots, n$ every player considers her best option (i.e., the outgoing edge of smallest color) in the current graph.
2. The considererd edges induce node-disjoint directed cycles and loops. Let N_i be the set of players that form these cycles in iteration i.
3. The algorithm reassigns the houses to the players in N_i consistently according to their preferences (according to the selected edges).
4. Before starting the next iteration, the algorithm removes the nodes corresponding to N_i (and their incident edges) from the graph and it increases i.

1. Apply the TTCA to the following instance with players a, b, c, d.

$$\begin{aligned} a &: \ b \succ c \succ a \succ d \\ b &: \ c \succ a \succ b \succ d \\ c &: \ d \succ a \succ c \succ b \\ d &: \ d \succ c \succ a \succ b \end{aligned}$$

2. Re-prove for this case that the outcome of the TTCA is indeed in the core of the House Allocation problem, that is, there is no blocking coalition S among the players for the allocation produced by TTCA. (Give a direct proof.)

3. Consider the following **modified version** of the TTCA (which is the algorithm given in the AGT book) where the last step is done as follows:

> 4*. Before starting the next iteration, the algorithm removes all the edges of color i and all players in N_i, and it increases i.

Does the output of this algorithm belong to the core of the House Allocation problem? Provide an argument or a counterexample.

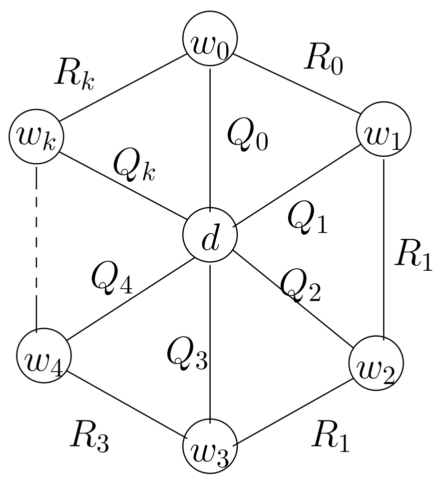
Next exercises are about Lecture 9

Exercise 2: (4 Points)

In this exercise we want to show the implication

$$\text{Gao-Rexford} \Rightarrow \text{No Dispute Wheel} \tag{1}$$

(See lecture notes on BGP for the definitions.)
Consider this kind of simpler wheels (paths R_i and Q_i consist of a single link):



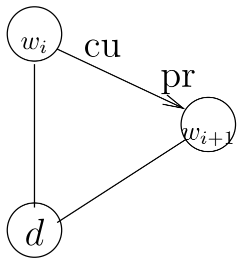
where the preferences of the nodes are

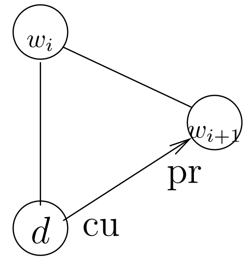
$$Q_i \prec_{w_i} R_i Q_{i+1} \tag{2}$$

Your task is:

1. Prove (1) for the simple wheels as above.
2. Discuss how to extend the proof to a general wheel.

Hint: Recall that $\emptyset$ denotes any path that does not allow w_i to reach d (in particular if w_{i+1} does not allow transit traffic from w_i) and the utility is $u_{w_i}(\emptyset) = 0$ (the lowest possible). Show that (2) is possible only in one of these two cases:





Exercise 3: (3 Points)

With this exercise we want to understand why we define the total utility (and incentive compatibility) as a 'lim sup' (see lecture notes).

Consider this NBR-solvable game given in the lecture notes:

	A	B	C
a	1 2	0 0	0 0
b	2 1	-1 1	1 -1
c	-2 -1	1 -1	-1 1

Prove that best-response dynamics are **not incentive compatible** (i.e., show a starting state and an activation sequence for which one player can improve his/her total utility by deviating from best-response).