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Deadline: Beginning of next lecture

Algorithmic Game Theory

Fall 2016

Exercise Set 11

Exercise 1: (2+1 Points)
In this exercise we want to show that the VCG mechanism for sponsored search satisfies the following two conditions:

- **Envy-freeness** meaning that no bidder getting slot s would like to get slot $s+1$ and pay the price of bidder $s+1$, nor slot $s-1$ and pay the price of bidder $s-1$:

$$\alpha_s v_s - P_s^{VCG}(v) \geq \alpha_t v_s - P_t^{VCG}(v) \qquad \text{for } t \in \{s-1, s+1\} \qquad (1)$$

- **Voluntary partecipation** which is the usual condition that truth-telling bidders have non-negative utilities.

Exercise 2: (4 Points)
In this exercise we want to show that symmetric pure Nash equilibria do exist. In particular, we want to prove this theorem stated in the lecture notes:

Theorem 10 *There exists always a symmetric pure Nash equilibrium whose revenue is the same as the revenue achieved by VCG on input the true valuations.*

Your task is to prove this theorem: for every valuations v, it is possible to construct bid vector b^{VCG} such that

$$P_s^{VCG}(v) = P_s^{GSP}(b^{VCG})$$

and b^{VCG} is a symmetric pure Nash equilibrium.
Hint: it might be useful to use the simpler characterization of SPE in the lecture notes.

Exercise 3: (4 Points)
A desirable property in sponsored search practice is that prices are decreasing with slots and higher slots have higher prices per click and in total:

$$\alpha_{s-1} p_{s-1} \geq \alpha_s p_s \qquad \text{and} \qquad p_{s-1} \geq p_s \qquad \text{for all } s. \qquad (2)$$

These conditions are clearly satisfied by the GSP mechanism using $p_s = b_{s+1}$.

Show that this remains valid if we consider symmetric pure Nash equilibria with respect to a generic mechanism charging bidder s an amount $\alpha_s \cdot p_s$ (derive (2) from the conditions of SNE without using $p_s = b_{s+1}$).