ETH Zürich
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Deadline: Beginning of next lecture

Algorithmic Game Theory

Fall 2019

Exercise Set 1

These exercises are **non-graded**. Please hand in your solutions at the beginning of next lecture (Sept 27) or by email (same deadline, to paolo.penna@inf.ethz.ch) to get feedbacks on your solutions.

Exercise 1: (1 Points)

Show that best response are not guaranteed to find pure Nash equilibria if they exist. That is, show a game such that (1) the game has a pure Nash equilibrium, but (2) best response on this game do not always terminate.

Exercise 2: (1 Points)

Consider the following weaker form of best response, called *better* response. A strategy $s'_i \in S_i$ is a *better* response to s if $u_i(s) < u_i(s'_i, s_{-i})$.

Show that there exist games in which (1) best response terminate always, but (2) better response do not terminate; that is, there exist a sequence of better response which cycles:

$$s^1, s^2, \ldots, s^k, s^{k+1} = s^1$$

such that each s^{l+1} is obtained from s^l by letting one player changing his/her strategy into a better response to s^l.

Exercise 3: (3 Points)

Consider an arbitrary game with *two* players in which best response are guaranteed to terminate from any starting state. Suppose further that in this game best response are always *unique*, that is, if $s_i \in S_i$ is a best response to s, then

$$u_i(s'_i, s_{-i}) < u_i(s^*_i, s_{-i})$$

for all $s'_i \in S_i$.

Prove that best response reach always an equilibrium in at most $O(m)$ steps, where m is the number of strategies of each player. (Note that $O(m^2)$ is trivial.)

Discuss how the assumption that best response are unique can be dropped and still obtain an algorithm which finds a pure Nash equilibrium in $O(m)$ steps.

Exercise 4: (3 Points)

In this exercise we ask you to prove a key part on the theorem regarding the convergence time in *Singleton Congestion Games*. Specifically, we have the following (see lecture notes):

1. Given the m resources, sort the set of delay values $V = \{d_r(k) \mid 1 \leq r \leq m, 1 \leq k \leq n\}$ in increasing order.

2. Define this alternative new delay functions: $\bar{d}_r(k) :=$ position of $d_r(k)$ in sorted list.

Your exercise is to prove the following:

> Suppose $s \to s' = (s'_i, s_{-i})$ is an improvement step for some player i with respect to the original delays. Then $s \to s' = (s'_i, s_{-i})$ is an improvement step for i with respect to the new delays as well.

Here 'improvement step' means exactly better response (exercise above).