ETH Zürich
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Deadline: Beginning of next lecture

Algorithmic Game Theory

Fall 2019

Exercise Set 3

Your solutions to this exercise sheet will be **graded**. Together with the other three graded exercise sheets, it will account for 30% of your final grade for the course. You are expected to solve the exercises carefully and then write a nice complete exposition of your solution (preferably using **LaTeX** or similar computer editors – the appearance of your solution will also be part of the grade). You are welcome to discuss the tasks with your fellow students, but we expect each of you to hand in **your own** individual write-up. Your write-up should list all collaborators. The deadline for handing in the solution is **Oct 11, before midnight**. Please send your PDF via email to paolo.penna@inf.ethz.ch. Your file should have the name <Surname>.pdf, where <Surname> should be exchanged with your family name.

Exercise 1: (1+3 Points)

In the lecture we have seen bounds on the Price of Anarchy for congestion games with *affine delay functions*, that is, when all delay functions are (non-decreasing) of the form

$$d_r(x) = a_r \cdot x + b_r,$$

with $a_r, b_r \geq 0$.

Now consider the following class of congestion games with *polynomial delay functions* of degree at most d,

$$d_r(x) = x^{d_r}$$

where $0 \leq d_r \leq d$.

Your task:

1. In the lecture we have seen that for affine delay functions,

   $$PoA_{\mathsf{PNE}} = PoA_{\mathsf{CCE}} = 5/2.$$

   Show that for some $d > 1$, $PoA_{\mathsf{PNE}} > 5/2$.

2. Show that, for every fixed $d \geq 1$, the Price of Anarchy is constant (even for coarse correlated equilibria):

   $$PoA_{\mathsf{CCE}} \leq C$$

   where C is a constant depending only on the maximum degree d.

   You can use the following result (without proving it):

> For every d there exist constants $c_1 < 1$ and c_2 such that
>
> $$y(x+1)^k \leq c_1 \cdot x^{k+1} + c_2 \cdot y^{k+1}$$
>
> for all $0 \leq k \leq d$ and for all positive integers x and y.

Note: If you cannot prove the bound for coarse correlate equilibria, try to prove the bound just for pure Nash equilibria.

Exercise 2: (3 Points)
Nash's Theorem states that every finite strategic game has a mixed Nash equilibrium. Prove this theorem for the special case in which we have only **two players** and each of them has only **two strategies**.

Exercise 3: (3 Points)
Prove that computing pure Nash equilibria in congestion games remains PLS-complete also when we restrict to **affine delay functions**. That is, for $d_r(x) = a_r x + b_r$ with $a_r, b_r \geq 0$. (Note that the condition $a_r \geq 0$ and $b_r \geq 0$ is important.)