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November 15, 2019

Deadline: Beginning of next lecture

Algorithmic Game Theory

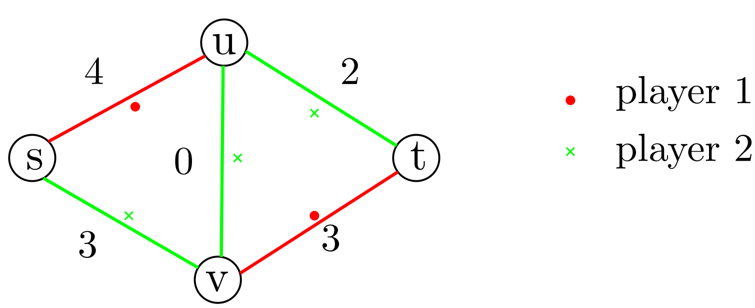
Fall 2019

Exercise Set 9

Your solutions to this exercise sheet will be **graded**. Together with the other three graded exercise sheets, it will account for 30% of your final grade for the course. You are expected to solve the exercises carefully and then write a nice complete exposition of your solution (preferably using **LaTeX** or similar computer editors – the appearance of your solution will also be part of the grade). You are welcome to discuss the tasks with your fellow students, but we expect each of you to hand in **your own** individual write-up. Your write-up should list all collaborators. The deadline for handing in the solution is **Nov 22, before 23:59**. Please send your PDF via email to paolo.penna@inf.ethz.ch. Your file should have the name <Surname>.pdf, where <Surname> should be exchanged with your family name.

Exercise 1: (2+1+3 Points)

Consider the following multi-parameter mechanism design problem. We are given a graph $G(V, E)$ and two nodes s and t, and each edge has a private cost which is known to the player owning this edge. We consider the case in which a player may own **several edges**. Here is an example with two players:



The feasible solutions consist of the paths connecting s to t, and the cost for a player is given by the sum of the costs of his/her edges in the chosen path. For example, the shortest path in the example above costs 0 to player 1 (red) and 5 to player 2 (green).

Here however we do not want to find the shortest path. We instead consider a different optimization goal, namely, we want to minimize the **maximum cost** among the players. In the example above, this would mean that we choose the lower path which results in a cost of 3 for both players (the maximum cost is 3 instead of 5).

Any solution a costs to player i

$$t_i(a) = \sum_{e \in E_i \cap a} t_i^e$$

where E_i is the subset of edges owned by i, a is a path (set of edges) connecting s to t, and t_i^e denotes the true cost of edge $e \in E_i$. We are interesting in minimizing the **maximum cost** among the players:

$$maxcost(a,t) := \max_i t_i(a)$$

Your task is:

1. Prove that no truthful mechanism (A,P) can minimize the maximum cost.
2. Give an n-approximate truthful mechanism, where n is the number of players.

   (An α-approximate mechanism guarantees, for every input t, $maxcost(A(t),t) \leq \alpha \cdot OPT(t)$, where $OPT(t) = \min_{a\in\mathcal{A}} maxcost(a,t)$ is the optimum maxcost.)
3. Strengthen your first result to show that for 2 players a truthful 2-approximation is the best possible.

   (You can solve this question directly and get also the points for the first one.)

Note: In this problem, each player i reports some cost c_i^e for each of his/her edges $e \in E_i$. Based on the reported costs c the mechanism selects a solution $A(c)$ and payments $P_i(c)$.

Exercise 2: (4 Points)

We consider the scenario in which there are **two players (voters)** and **at least three alternatives** (i.e., $\mathcal{A}$ is the set of alternatives with finite cardinality $|\mathcal{A}| \geq 3$). Recall that a **social choice function** f maps each profile P into a **single outcome** (winning alternative):

$$f(P) \in \mathcal{A}$$

for every $P = (P_1, P_2)$, where P_i is a preference order of player i over $\mathcal{A}$.

Natural conditions on f

1. **No alternative discarded:** For every $a \in \mathcal{A}$, there is a profile $P = (P_1, P_2)$ such that $f(P) = a$.
2. **Truthful:** Reporting the true preference order is dominant for both players.
3. **Unanimity:** If both players have the same top ranked alternative, this alternative must be the result. That is [a]

$$top(P_1) = a = top(P_2) \Rightarrow f(P) = a.$$

[a] Here $top(P_i) \in \mathcal{A}$ is the top preference according to preference order P_i over $\mathcal{A}$.

Your task is to prove the following implication:

- If f is truthful and no alternative is discarded, then f must satisfy unanimity.

Note: In this problem, each P_i is a total order over $\mathcal{A}$, that is, for any two alternatives $a, b \in \mathcal{A}$, either

$$aP_ib \quad (\text{"prefer } a \text{ over } b\text{"}) \qquad \textit{or} \qquad bP_ia \quad (\text{"prefer } b \text{ over } a\text{"}) \tag{1}$$

A social choice function is **not truthful** if there exists a profile P, a player i, and some P_i' such that

$$a'P_ia$$

where $a = f(P)$ and $a' = f(P_i', P_{-i})$. As usual, (P_i', P_{-i}) is the profile obtained from P by changing P_i into P_i'.

Exercise 3: (2+2 Points)

Consider the **House Allocation** problem with n players and the following algorithm that is equivalent to the **Top Trading Cycle algorithm (TTCA)** presented in the lecture. The algorithm operates on the following (complete) directed graph:

- Every player i (and his/her house) is represented by a vertex i;
- If house j is player i's k^{th} choice, we add a directed edge (i, j) of color k.

The algorithm works as follows:

TTCA:

1. In every iteration $i = 1, \ldots, n$ every player considers her best option (i.e., the outgoing edge of smallest color) in the current graph.
2. The considererd edges induce node-disjoint directed cycles and loops. Let N_i be the set of players that form these cycles in iteration i.
3. The algorithm reassigns the houses to the players in N_i consistently according to their preferences (according to the selected edges).
4. Before starting the next iteration, the algorithm removes the nodes corresponding to N_i (and their incident edges) from the graph and it increases i.

Apply the TTCA to the following instance with players a, b, c, d:

$$\begin{aligned}
a &: \ b \succ c \succ a \succ d \\
b &: \ c \succ a \succ b \succ d \\
c &: \ d \succ a \succ c \succ b \\
d &: \ d \succ c \succ a \succ b
\end{aligned}$$

Your task:

1. Re-prove for this case that the outcome of the TTCA is indeed in the core of the House Allocation problem, that is, there is no blocking coalition S among the players for the allocation produced by TTCA. (Give a direct proof.)

2. Consider the following **modified version** of the TTCA (which is the algorithm given in the AGT book) where the last step is done as follows:

> 4*. Before starting the next iteration, the algorithm removes all the edges of color i and all players in N_i, and it increases i.

Does the output of this algorithm belong to the core of the House Allocation problem? Provide an argument or a counterexample.