Algorithmic Game Theory | Fall 2019, Week 1

Strategic games, existence and convergence to equilibria, congestion games

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1 Strategic Games

These are the three games we discussed during the lecture. The numbers represent the utility (payoff) that each player gets given the strategies chosen by the two players (row player and column player). The bottom-left number is the utility of row player, and upper-left number is the utility of column player.

	H	T
H	-1 1	1 -1
T	1 -1	-1 1

Matching Pennies

	B	S
B	1 2	0 0
S	0 0	2 1

Battle of Sexes

	A	S
A	-2 -2	-3 0
S	0 -3	-1 -1

Prisoners' Dilemma

> **Best Response (informal)**: Players move in turns (alternate), and each player tries to maximize his/her own utility (given the other player current choice).

What happens for the three games above?

Strategic Game:

1. A set of n players denoted as $\{1, 2, \ldots, n\}$.
2. Each player i has
   (a) A set S_i of possible strategies;
   (b) A utility function $u_i()$;

When each player i chooses some $s_i \in S_i$, the resulting combination

$$s = (s_1, \ldots, s_i, \ldots, s_n)$$

yields the utility of each player, that is

$$u_i(s) = u_i(s_1, \ldots, s_i, \ldots, s_n).$$

The possible states are $S = S_1 \times S_2 \times \cdots \times S_n$.

Each player i can only decide/change her own strategy s_i. The goal of player i is to maximize her own utility given the choices of the others.

Notation: Given a combination of strategies

$$s = (s_1, \ldots, s_i, \ldots, s_n)$$

we distinguish between the choices of players *other than i*

$$s_{-i} := (s_1, \ldots, s_{-i}, *, s_{i+1}, \ldots, s_n)$$

and the vectors in which only player i changes strategy in s,

$$(s'_i, s_{-i}) := (s_1, \ldots, s_{i-1}, s'_i, s_{i+1}, \ldots, s_n).$$

The next two definitions capture how/when players change their strategies, and if they eventually decide to 'rest' on a particular state of the game.

(Pure Nash) Equilibrium: A state s^* in which no player has incentive to move:

$$u_i(s^*) \geq u_i(s'_i, s^*_{-i}), \quad \text{for every player } i, \text{ and for all } s'_i \in S_i. \tag{1}$$

Definition 1 (best response). *A strategy $s^*_i \in S_i$ is a best response to s if*

$$u_i(s^*_i, s_{-i}) = \max_{s'_i \in S_i} u_i(s'_i, s_{-i}).$$

Best Response Algorithm

1. If in the current state s one or more players can strictly improve then
   (a) Pick one of these players (say player i);
   (b) Make the player switch to a best response:
   $$s \to s' = (s'_i, s_{-i})$$
   where i is the player and s'_i is a best response to s.
   (c) Restart (go to Step 1) with the new state $s' = (s'_i, s_{i-1})$.
2. Else terminate.

We say that **best response converge** if, for any initial state s, the best response algorithm[1] above terminates.

Exercise 1. *Show that, if best response algorithm terminates in some state s, then this state must be a (pure Nash) equilibrium.*

2 Congestion and Potential Games

Potential Game: A strategic game is a potential game if there exists a function $f()$ such that
$$u_i(s) - u_i(s'_i, s_{-i}) = f(s) - f(s'_i, s_{-i}) \quad (2)$$
for all i, for all $s \in S$, and for all $s'_i \in S_i$.

The intuition behind this definition is that, every time one player improves his/her utility, the function $f()$ changes accordingly.[2]

Exercise 2. *Show that the 'Battle of Sexes' game above is a potential game.*

Theorem 2. *Best response converge to a (pure Nash) equilibrium in every potential game with finitely many strategies.*[3]

Proof. By contradiction, suppose best response cycles:
$$s^1 \to s^2 \to \cdots \to s^k \to s^1$$
where '$\to$' denotes a best response of some player (that is, $s \to s'$ means that $s' = (s^*_i, s_{-i})$ and s^*_i is a best response to s, for some i). Therefore, by (2)
$$f(s^1) < f(s^2) < \cdots < f(s^k) < f(s^1).$$
Starting from any initial state, best response terminates to some state s^t. This state must be a (pure Nash) equilibrium (**Exercise 1**). □

[1] This algorithm is usually called *Best Response Dynamics.*
[2] Games satisfying (2) are usually called *Exact Potential Games.*
[3] Each player has a finite set S_i of strategies.

Exercise 3. *Let us call* social welfare *the sum of all players' utilities:*

$$SW(s) := \sum_i u_i(s).$$

I claim that every time a player plays a best response in a potential game, the social welfare improves (increases). Disprove my claim: Show a potential game in which there is a best response s_i^* to some s such that $SW(s) > SW(s_i^*, s_{-i})$.

Congestion Games:

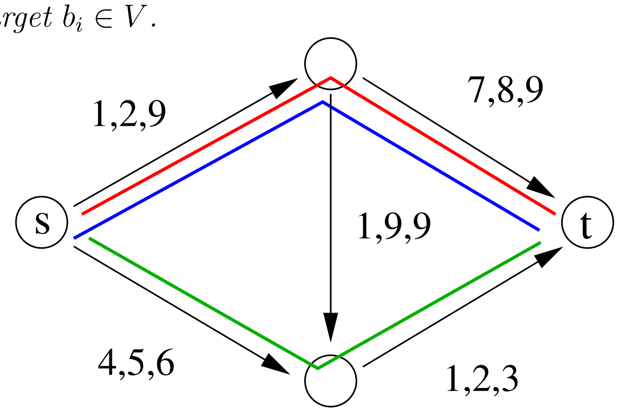
1. We have n players and m **resources**.
2. Each player can choose among some subsets of resources:
$$S_i = \{\ldots, s_i, \ldots\}, \qquad s_i \subseteq \{1, \ldots, m\}$$
3. Each resource r has a delay function
$$d_r(x)$$
4. Cost of player i is
$$c_i(s) := \sum_{r \in s_i} d_r(n_r(s))$$
where $n_r(s)$ is the number of players using resource r in s.[a]

Players want to minimize their costs, so we can say "utility = - cost",

$$u_i(s) = -c_i(s).$$

[a] $n_r(s) = |\{i | r \in s_i\}|$

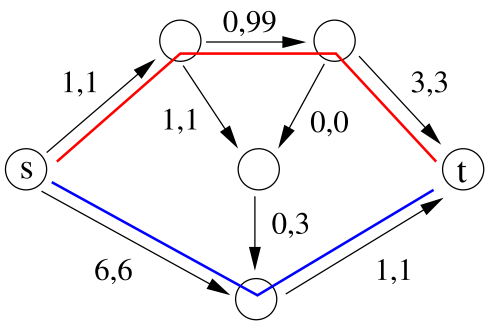
Example 3 (Network Congestion Game)**.** *Given a directed graph $G = (V, E)$ with delay functions $d_e \colon \{1, \ldots, n\} \to \mathbb{Z}$, $e \in E$. Player i wants a path of minimal delay from a source $a_i \in V$ to a target $b_i \in V$.*



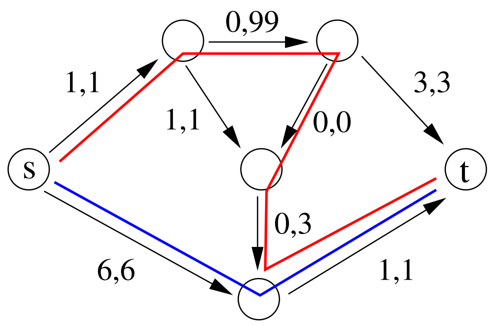
In this example, all three players want to go from node s *to node* t*, meaning that S_i = 'set of* s-t *paths'.*

Example 4. *A sequence of best response steps:*

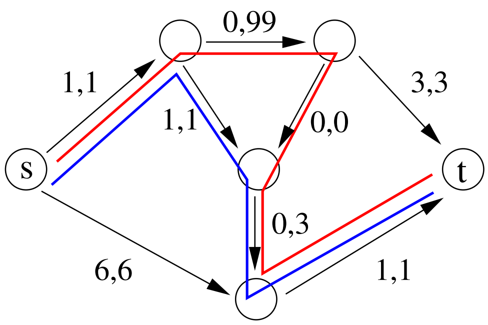
start:



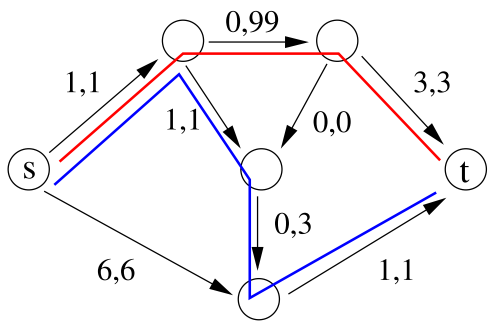
after 1^{th} *best response (red player):*



after 2^{nd} *best response (blue player):*



after 3^{nd} *best response (red player):*



reached pure Nash equilibrium

Questions

- Does every congestion game posses a pure Nash equilibrium?
- Do best response converge?
- How many steps does it take?

Theorem 5. *Congestion games are potential games. Therefore best response converge to a (pure Nash) equilibrium in finitely many steps.*

Proof. We show that (2) is satisfied by considering the following function:

$$\Phi(s) := \sum_{r=1}^{m} \sum_{k=1}^{n_r(s)} d_r(k) \quad . \tag{3}$$

This function is called *Rosenthal's potential function.*

Lemma 6. *Let s be any state. Suppose we go from s to a state s' by an improvement step of player i decreasing his delay by $\Delta > 0$,*

$$c_i(s) - c_i(s') = \Delta \quad .$$

Then $\Phi(s) - \Phi(s') = \Delta$.

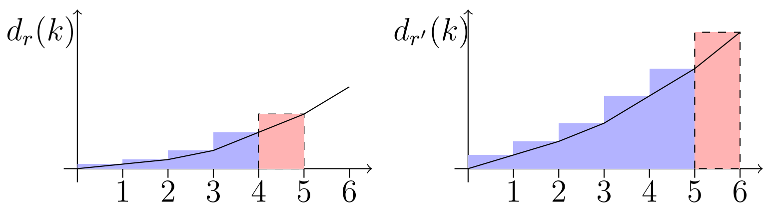


Figure 1: Proof of Lemma 6: The contribution of two resources r and r' to the potential is the shaded area. If a player changes from r' to r, his delay changes exactly as the potential value (difference of red areas).

Proof. The potential $\Phi(s)$ can be calculated by inserting the players one after the other in any order, and summing the delays of the players at the point of time at their insertion.

Without loss of generality player i is the last player that we insert when calculating $\Phi(s)$. Then the contribution of player i to the potential is just delay of player i in state s. When going from s to s', the delay of i decreases by Δ, and, hence, Φ decreases by Δ as well (see Figure 1 for an example.) □

The lemma shows that (2) is satisfied (simply take $f = -\Phi$), that is, every congestion game is a potential game. □

3 Convergence Time of Best Response

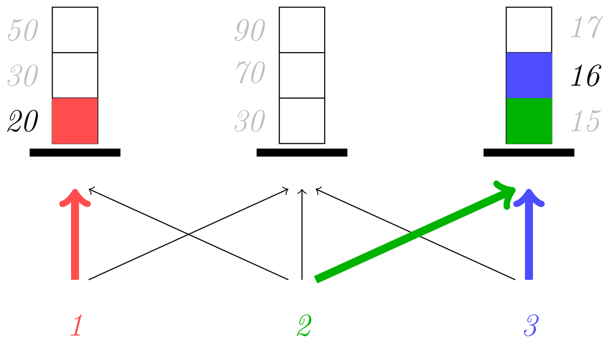
In general, if best response converge, then it will visit each state at most once. This means that it reaches an equilibrium in at most N steps, where N denotes the number of possible states ($N = |S|$ for $S = S_1 \times S_2 \times \cdots \times S_n$). In congestion games, the bound we have with n players and m resources is 2^{mn}, which in general not the best possible.

We will show a significantly better, namely *polynomial*, bound for *singleton congestion games*. In this subclass of congestion games every player wants to allocate only a *single resource* at a time from a subset of allowed resources. Formally:

Definition 7 (Singleton Congestion Games)**.** *A congestion game is called* singleton *if, for every player i and every strategy $s_i \in S_i$, it holds that $|s_i| = 1$.*

Although this constraint on the strategy sets is quite restrictive, there are still up to m^n different states.

Example 8 (Singleton Congestion Game)**.** *Consider a "server farm" with three servers a, b, c (resources) and three players 1,2,3 each of which wants to access a single server.*



The colored arrows indicate a pure Nash equilibrium.

Theorem 9. *In a singleton congestion game with n players and m resources, best response converge in at most $O(n^2\,m^2)$ steps.*

Proof idea:

- Replace original delays by bounded integer values without changing the preferences of the players.
- Show an upper bound on the maximum potential with respect to new delays.
- Due to integer values, decrease of potential in an improvement step is at least 1. Hence, length of every improvement sequence is bounded by maximum potential.

Proof. Sort the set of delay values $V = \{d_r(k) \mid 1 \le r \le m, 1 \le k \le n\}$ in increasing order. Define alternative, new delay functions:

$$\bar{d}_r(k) := \text{position of } d_r(k) \text{ in sorted list.}$$

The new delay of a player i using resource r in state s is just $\bar{d}_r(n_r(s))$.

Observation 10 (Exercise!). *Let s and $s' = (s'_i, s_{-i})$ be two states such that $s \to s'$ is an improvement step for some player i with respect to the original delays. Then $s \to s' = (s'_i, s_{-i})$ is an improvement step for i with respect to the new delays, as well.*

Here 'improvement step' means that the cost of player i in s' is strictly smaller than his/her cost in s.

Furthermore, observe that $\bar{d}_r(k) \le nm$ for all $r \in [m]$ and $k \in [n]$ because there are at most nm elements in V. Therefore, Rosenthal's potential function (3) with respect to the new delays $\bar{d}_r(k)$ can be upper-bounded as follows:

$$\bar{\Phi}(s) = \sum_{r=1}^{m} \sum_{k=1}^{n_r(s)} \bar{d}_r(k) \le \sum_{r=1}^{m} \sum_{k=1}^{n_r(s)} n\,m \le (n\,m)^2 \ .$$

It holds that $\bar{\Phi} \ge 1$. Also, $\bar{\Phi}$ decreases by at least 1 in every step. Therefore, the length of every improvement sequence is upper-bounded by $(n\,m)^2$. □

Exercise 4. *I am not very clever in the upper bound above. Look at it again and show that $O(n^2\,m)$ is the correct bound.*

Example 11. *The sorted list of delay values in Example 8 is*

$$15, 16, 17, 20, 30, 50, 70, 90.$$

Hence, the old and new delay functions are

$$d_a(1,2,3) = (20,30,50) \qquad \bar{d}_a(1,2,3) = (4,5,6)$$
$$d_b(1,2,3) = (30,70,90) \qquad \bar{d}_b(1,2,3) = (5,7,8)$$
$$d_c(1,2,3) = (15,16,17) \qquad \bar{d}_c(1,2,3) = (1,2,3)$$

Recommended Literature

- D. Monderer, L. Shapley. Potential Games. Games and Economic Behavior, 14:1124–1143, 1996. (Equivalence congestion and potential games)
- H. Ackermann, H. Röglin, B. Vöcking. On the impact of combinatorial structure on congestion games. Journal of the ACM, 55(6), 2008. (Generalization of theorem on singleton games)

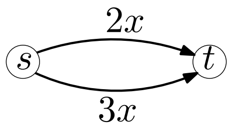
A significant part of this notes is from previous years' notes by Paul Dütting available here:

- http://www.cadmo.ethz.ch/education/lectures/HS15/agt_HS2015

Exercises
(during next exercise class - 24.9.2019)

We shall discuss and solve together the following exercises:

Exercise 5. *We have these two parallel links with linear delay functions:*



For two players, write the corresponding game matrix (payoffs). By looking at this matrix, find a potential function $f(\cdot)$ *satisfying* (2).

Exercise 6. *I want to use a simpler potential function. I propose this one, which is the sum of the delays of all resources:*

$$\tilde{\Phi}(s) := \sum_{r=1}^{m} d_r(n_r(s)) \ .$$

I claim that

- *This function* $\tilde{\Phi}(\cdot)$ *satisfies the definition of potential function* (2).
- *If we find an* s *minimizing* $\tilde{\Phi}(\cdot)$ *then* s *is a Nash equilibrium.*

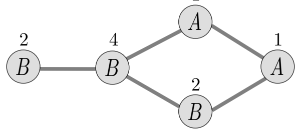
Both claims are ***wrong****. Say why by providing a counterexample.*

Exercise 7. *Two players can choose between* two technologies *and the payoff is related to the following game:*

	A	B
A	1, 1	0, 0
B	0, 0	2, 2

The game says that (for compatibility reasons) it is better to choose the same technology, though technology B *is superior to technology* A.

We have a social network (a graph) where each node is a player and an edge represents a friendship (or collaboration). The players play the game above with each of their friend (neighbors) and they must choose one of the two technologies. Here is an example:



1. *Each player i chooses one of the two technologies;*
2. *The payoff of player i is the sum of the payoffs of the two-player game above, for each of the friends (neighbors) of i.*

Show that best response always converge to a Nash equilibrium.