Sponsored Search and (Non-)Truthful Mechanisms

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In this lecture we introduce the Sponsored Search Auction problem, essentially how current search engine sell slots of their pages. We shall see and compare two approaches:

- The VCG payment scheme;
- The GSP (Generalized Second Price auction) which is used by current search engines (Google, Yahoo, etc.).

Why VCG is not preferred to GSP?

In this lecture we study this problem and try to answer this question. This analysis is based on the Price of Anarchy and Stability of in the game resulting from GSP pricing schemes.

1 Sponsored Search Auctions

We begin by defining sponsored search auctions, and how the VCG mechanism for this setting looks like. We then define the GSP mechanism, which is used by most search engines in practice.

Definition 1 (Sponsored Search Auction)**.** *There are n bidders competing for the assignment of one of k slots. Each bidder i has a (private) value-per-click v_i and each slot j has a (known) click-through-rate α_j. We assume that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$. Bidder i's value for slot j is given by the product $\alpha_j \cdot v_i$.*

Definition 2 (VCG Mechanism)**.** *The Vickrey-Clarke-Groves Mechanism for sponsored search auctions proceeds as follows:*

1. *Collect a bid b_i from each agent $i \in \{1, \ldots, n\}$.*
2. *Sort bidders such that $b_1 \geq b_2 \geq \cdots \geq b_n$.*
3. *For $i = 1, \ldots, k$: Assign bidder i to slot i and make him/her pay*

$$P_i^{VCG}(b) := \sum_{\ell=i}^{k} (\alpha_\ell - \alpha_{\ell+1}) \cdot b_{\ell+1} \tag{1}$$

where $\alpha_{k+1} = 0$ represents a "non-existing" slot.

Definition 3 (GSP Mechanism)**.** *The Generalized Second-Price Mechanism for sponsored search auctions proceeds as follows:*

1. *Collect a bid b_i from each agent $i \in \{1, \ldots, n\}$.*
2. *Sort bidders such that $b_1 \geq b_2 \geq \cdots \geq b_n$.*
3. *For $i = 1, \ldots, k$: Assign bidder i to slot i and make him/her pay*

$$P_i(b) := \alpha_i \cdot b_{i+1} \quad . \tag{2}$$

Whenever bidders are truth-telling ($b_i = v_i$) both mechanisms maximize the **social welfare**

$$SW(b) = \sum_{i=1}^{k} \alpha_i \cdot v_i,$$

and the only difference is on the payments part.

GSP
Simpler + Higher Revenue + Not Truthful

Note that GSP uses **per-click payments** (2), charging $p_s = b_{s+1}$ per click to the bidder getting slot s (recall we renamed bidders so that s is the bidder with s^{th} highest bid). The **revenue** of the search engine is the sum of the payments received by the bidders

$$R(b) = \sum_{i=1}^{k} P_i(b).$$

It turns out the the revenue of GSP is always higher than that of VCG:

$$P_s^{GSP}(b) \geq P_s^{VCG}(b) \qquad \text{for all } b \text{ and all slots } s. \tag{3}$$

Exercise 1. *Prove* (3).

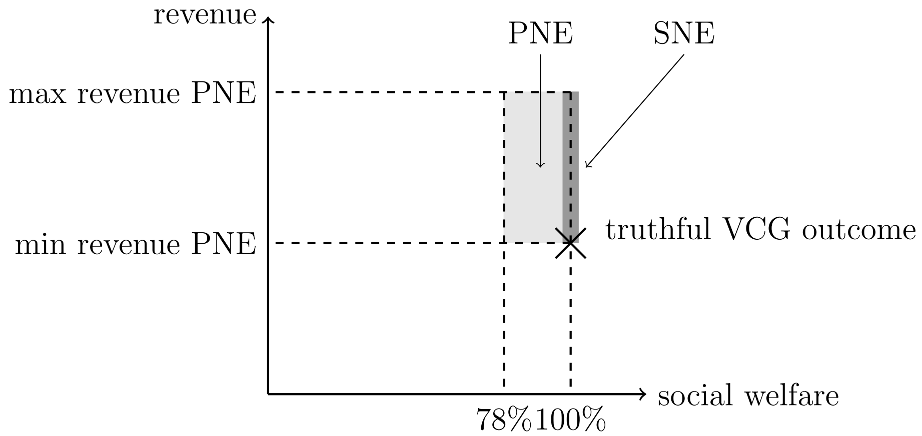


Figure 1: Overview of results. Pure Nash equilibria are light gray, while symmetric pure Nash equilibria are dark gray.

The GSP mechanism (unlike the VCG mechanism) is **not truthful**. Recall that the utility of bidder i in a given mechanism using payments P is

$$u_i(b) = \alpha_j \cdot v_i - P_i(b) \ ,$$

where j is the slot assigned to i for bids b.

Example 4 (GSP is not truthful)**.** *There are three bidders and three slots. The click-through rates are $\alpha = (1, 0.7, 0.1)$ and the valuations are $v = (20, 10, 5)$. Suppose the bidder with valuation 20 faces bids 10 and 5. A truthful bid gives him/her slot 1 for a utility of $1.0 \cdot (20 - 10) = 10$. Any bid between 10 and 5 would give him/her slot 2 for a utility of $0.7 \cdot (20 - 5) = 10.5$.*

Because bidders can gain by misreporting their valuations, we analyze GSP in terms of induced (pure) Nash equilibria.

2 Equilibria in the GSP Mechanism

In a pure Nash equilibrium a bidder that is assigned slot s prefers this slot over any slot $t < s$ or $t > s$. We obtain the following characterization in terms of the per click payments $p_s = b_{s+1}$ in the GSP mechanism.

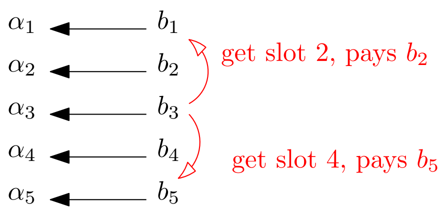
Observation 5. *A bid profile b is a pure Nash equilibrium (PNE) if and only if for all bidders s,*

$$\alpha_s \cdot (v_s - p_s) \geq \alpha_t \cdot (v_s - p_t) \qquad \text{for all } t > s, \text{ and} \tag{4}$$

$$\alpha_s \cdot (v_s - p_s) \geq \alpha_t \cdot (v_s - p_{t-1}) \qquad \text{for all } t < s, \tag{5}$$

where we omitted the bids b from the formulas for notational convenience.

Remark 1 (asymmetry)**.** *Note the asymmetry in this definition that stems from the fact that if bidder s targets a better slot $t < s$ then he/she has to "jump" in front of b_t and pay $b_t = p_{t-1}$:*



If instead bidder s targets a worse slot $t > s$, then he/she has to "jump" right after t, this bidder will get slot $t - 1$, and bidder s pays $b_{t+1} = p_t$.

It turns out that there is always a pure Nash equilibrium of the GSP mechanism, which maximizes the social welfare and in which every bidder pays what he/she would pay in the truthful VCG equilibrium.

Example 6 (optimal PNE)**.** *Consider the same instance of Example 4. Suppose the bidder with valuation* 20 *bids* 10*, the bidder with valuation* 10 *bids* 6*, and the bidder with valuation* 5 *bids* $30/7 \approx 4.286$*. Then the highest valuation bidder wins the first slot at a price of* 6*, the second highest valuation bidder wins the second slot at a price of* 3*, and the third and lowest value bidder wins the third slot for free.*

Not all pure Nash equilibria induced by the GSP mechanism maximize the social welfare as the following example shows.

Example 7 (sub-optimal PNE)**.** *Suppose the bidder with value* 20 *bids* 6*, the bidder with value* 10 *bids* 10*, and the bidder with value* 5 *bids* $30/7 \approx 4.26$*. Then the second highest value bidder wins the first slot, the highest value bidder wins the second slot, and the third and lowest value bidder wins the third slot. Note that the social welfare is* 24.5*, which is a factor* ≈ 1.127 *smaller than the optimal social welfare of* 27.5*.*

We are interested in comparing the worst (and the best) pure Nash equilibrium to the optimal social welfare. That is, we consider the Price of Anarchy and the Price of Stability for our welfare-maximization problem:

$$PoA_{\mathsf{PNE}} = \frac{\max_{s \in S} SW(s)}{\min_{s \in PNE} SW(b)} \ , \qquad\qquad PoS_{\mathsf{PNE}} = \frac{\max_{s \in S} SW(s)}{\max_{s \in PNE} SW(b)} \ .$$

Example 7 says that $PoA_{\mathsf{PNE}} \geq 1.127$, and this cursory analysis is not too far off from the truth (see Figure 1). The Price of Anarchy with respect to pure Nash equilibria is known to be at most 1.282 and the Price of Stability is 1.[1] Instead of proving these results we will focus on a *refinement* of pure Nash equilibria—symmetric pure Nash equilibria—and show that for these equilibria $PoA = PoS = 1$. We will also show that the smallest revenue in any such equilibrium coincides with the revenue in the truthful VCG equlibrium and the largest revenue coincides with the maximum revenue in any pure Nash equilibrium.

2.1 Symmetric (Pure) Nash Equilibria

Symmetric pure Nash equilibria are obtained from pure Nash equilibria by removing the "asymmetry" in the equilibrium conditions (4)-(5). This removal in fact strengthens the equilibrium concept and so every symmetric pure Nash equilibrium will be a pure Nash equilibrium. The following definition requires that no bidder wants to **swap bids** with a different bidder (equivalently, no bidder s is envious of bidder t slot and price).

Definition 8 (Varian, 2007)**.** *A bid profile b is a symmetric pure Nash (SNE) equilibrium if for all bidders s,*

$$\alpha_s \cdot (v_s - p_s) \geq \alpha_t \cdot (v_s - p_t) \qquad\qquad \textit{for all } t,$$

where we again omitted the bids b from the formula for notational convenience.

[1] The Price of Anarchy bound for pure Nash equilibria is almost tight. There is an example with three bidders in which the Price of Anarchy is 1.259. For more general solution concepts such as coarse correlated equilibria the bound is slightly worse but still constant.

Symmetric Nash Equilibria
Always Exist + Maximize Social Welfare + Lowest Revenue as good as VCG

We first show the optimality of social welfare.

Theorem 9. *Every SNE maximizes the social welfare.*

Proof. It suffices to show that $v_s \geq v_{s+1}$ for all s such that $\alpha_s > \alpha_{s+1}$. From the definition of SNE, we get that for all bidders s it holds that[2]

$$(\alpha_{s+1} - \alpha_s) \cdot v_{s+1} \geq \alpha_{s+1} \cdot p_{s+1} - \alpha_s \cdot p_s, \quad \text{and} \tag{6}$$

$$(\alpha_s - \alpha_{s+1}) \cdot v_s \geq \alpha_s \cdot p_s - \alpha_{s+1} \cdot p_{s+1}. \tag{7}$$

By adding these inequalities we obtain

$$(\alpha_s - \alpha_{s+1})(v_s - v_{s+1}) \geq 0,$$

which shows that $(v_t)_t$ and $(\alpha_t)_t$ must be ordered the same way. □

Theorem 10. *There exists always a SNE whose revenue is the same as the revenue achieved by VCG on input the true valuations.*

Proof. For every valuations v, it is possible to construct bid vector b^{VCG} such that

$$P_s^{VCG}(v) = P_s^{GSP}(b^{VCG})$$

and b^{VCG} is a SNE (**Exercise!**). □

2.2 A Simpler Characterization of SNE

It turns out that SNE are equivalent to the following simpler condition. Instead of requiring that the inequalities hold for deviations to all possible slots it is sufficient to prevent deviations to the slot right above and right below.

Proposition 11 (characterization)**.** *If bids b satisfy the following two inequalities for all bidders s,*

$$\alpha_s \cdot (v_s - p_s) \geq \alpha_{s-1} \cdot (v_s - p_{s-1}), \quad \textit{and} \tag{8}$$

$$\alpha_s \cdot (v_s - p_s) \geq \alpha_{s+1} \cdot (v_s - p_{s+1}) \tag{9}$$

then these bids are a SNE.

Proof Idea. Instead of proving this result formally we will give the basic idea by considering an example with three bidders and three slots. The only "long haul" deviations that are not covered are deviations from 1 to 3 and from 3 to 1. We will show that bidder 1 will not find it beneficial to deviate to slot 3. The argument for the opposite direction is similar.

[2] These are the conditions that (1) bidder with slot s does not want slot $s + 1$ and (2) bidder with slot $s + 1$ does not want slot s.

We will first argue that $v_1 \geq v_2$. This is because the conditions (8)-(9) are equivalent to the conditions (6)-(7) above from which we concluded that the valuations are ordered according to the slots ($v_s \geq v_{s+1}$).[3]

Next we will use the fact that bidder 1 does not want to deviate to slot 2 and bidder 2 does not want to deviate to slot 3 together with the fact that $v_1 \geq v_2$ to conclude that bidder 1 does not want to deviate to slot 3. Namely,

$$\begin{aligned} \alpha_1 v_1 - \alpha_1 p_1 \geq \alpha_2 v_1 - \alpha_2 p_2 \quad &\Rightarrow \quad (\alpha_1 - \alpha_2) v_1 \geq \alpha_1 p_1 - \alpha_2 p_2 \\ \alpha_2 v_2 - \alpha_2 p_2 \geq \alpha_3 v_2 - \alpha_3 p_3 \quad &\Rightarrow \quad (\alpha_2 - \alpha_3) v_1 \geq \alpha_2 p_2 - \alpha_3 p_3, \end{aligned}$$

If we add up these two inequalities we obtain

$$(\alpha_1 - \alpha_3) v_1 \geq \alpha_1 p_1 - \alpha_3 p_3 \quad \Rightarrow \quad \alpha_1 v_1 - \alpha_1 p_1 \geq \alpha_3 v_1 - \alpha_3 p_3.$$

□

2.3 Bounds on the GSP prices for SNE

Since the agent in position s does not want to move down one slot and the agent in position $s+1$ does not want to move up one slot, we get conditions (6)-(7) which we rewrite here for convenience:

$$\begin{aligned} &\alpha_s (v_s - p_s) \geq \alpha_{s+1} (v_s - p_{s+1}), \qquad \text{and} \\ &\alpha_{s+1} (v_{s+1} - p_{s+1}) \geq \alpha_s (v_{s+1} - p_s). \end{aligned}$$

Combining these inequalities we obtain

$$(\alpha_s - \alpha_{s+1}) v_s + \alpha_{s+1} p_{s+1} \geq \alpha_s p_s \geq (\alpha_s - \alpha_{s+1}) v_{s+1} + \alpha_{s+1} p_{s+1}. \tag{10}$$

Recalling that $p_s = b_{s+1}$ we obtain

$$(\alpha_{s-1} - \alpha_s) v_{s-1} + \alpha_s b_{s+1} \geq \alpha_{s-1} b_s \geq (\alpha_{s-1} - \alpha_s) v_s + \alpha_s b_{s+1}. \tag{11}$$

These inequalities give equivalent characterizations of the equilibrium. Symmetric pure Nash equilibria can be found by recursively choosing a sequence of bids that satisfy these inequalities.

3 Revenue Guarantees for Symmetric Equilibria

Next we use the characterization of the equilibrium bids to obtain the SNEs with the lowest and highest revenue. Note a similarity between VCG prices (1) and (10):

$$\begin{aligned} P_s^{VCG}(b) =& (\alpha_s - \alpha_{s+1}) b_{s+1} + P_{s+1}^{VCG}(b) \ , \\ (\alpha_s - \alpha_{s+1}) v_s + P_{s+1}^{GSP}(b) \geq P_s^{GSP}(b) \geq& (\alpha_s - \alpha_{s+1}) v_{s+1} + P_{s+1}^{GSP}(b) \ . \end{aligned}$$

[3] Alternatively, you can redo the same proof: Since bidder 1 does not want to deviate to slot 2 and bidder 2 does not want to deviate to slot 1: $\alpha_1 \cdot (v_1 - p_1) \geq \alpha_2 \cdot (v_1 - p_2)$ and $\alpha_2 \cdot (v_2 - p_2) \geq \alpha_1 \cdot (v_2 - p_1)$ Adding these two inequalities we obtain $(\alpha_1 - \alpha_2)(v_1 - v_2) \geq 0$ which shows the claim.

Theorem 12. *The lowest revenue SNE yields the same revenue as the truthful VCG equilibrium.*

Proof. We obtain the revenue for the lowest revenue equilibrium b^L by considering the lower bound given by (11):

$$\alpha_{s-1}b_s^L \geq (\alpha_{s-1}-\alpha_s)v_s + \alpha_s b_{s+1}^L$$

and by reapplying (11) to the last term

$$\begin{aligned}&\geq (\alpha_{s-1}-\alpha_s)v_s + (\alpha_s - \alpha_{s+1})v_{s+1} + \alpha_{s+1}b_{s+2}^L \\ &\vdots \\ &= \sum_{t=s}^{k}(\alpha_{t-1}-\alpha_t)v_t.\end{aligned}$$

The proof is concluded by observing that $\alpha_{s-1}b_s^L$ is the GSP price for slot $s-1$, while the summation is the truthful VCG price for slot $s-1$. □

Theorem 13. *The highest revenue SNE yields the same revenue as the highest revenue Nash equilibrium.*

Proof. Since every SNE is also a PNE, the highest revenue in any PNE can only be higher than the highest revenue in any SNE. So to show equality, it suffices to show that the highest revenue in a SNE is at least as high as the highest revenue in a PNE.

To obtain the highest possible revenue in a SNE we again consider the recursive characterization of equilibrium bids, each time choosing the highest possible bid. If we start from (11) we thus get

$$\alpha_{s-1}b_s^U = (\alpha_{s-1}-\alpha_s)v_{s-1} + \alpha_s b_{s+1}^U.$$

Using $p_s = b_{s+1}$,

$$\alpha_s p_s^U = (\alpha_s - \alpha_{s+1})v_s + \alpha_{s+1}p_{s+1}^U.$$

On the other hand we can get a similar recursive formulation for the set of pure Nash equilibria. Namely, also in a PNE the bidder in slot s does not want to deviate and target the next lower slot $s+1$. So,

$$\alpha_s p_s^N \leq (\alpha_s - \alpha_{s+1})v_s + \alpha_{s+1}p_{s+1}^N.$$

Both recursions start at the bottommost slot $s = k$. Since the slot right below does not exist we can set $\alpha_{k+1} = 0$ and obtain

$$p_k^N \leq v_k = p_s^U.$$

Inspecting the above recursions we see that $p_s^U \geq p_s^N$ for all s. □

4 Final Remarks

Despite GSP is not truthful, the results on symmetric pure Nash equilibria provide a theoretical justification for its use in practice:

1. The social welfare is optimal;
2. The revenue is never below that of VCG;
3. The highest revenue is as good as the highest revenue is any (non-symmetric) equilibrium.

As long as bidders can find such an equilibrium, GSP are more appealing than VCG from the perspective of search engine. Also, the payments in GSP are simpler than those in VCG from the point of view of the bidders. The best-response mechanism framework in the previous lecture can be adapted to GSP auctions to prove that by repeatedly best-responding (1) bidders can compute an equilibrium corresponding to the truthful VCG outcome and (2) repeatedly best-responding is incentive compatible (Nisan et al. 2011). GSP is also preferable to VCG when α_i's are an estimate the true quality of the slots, as GSP is more robust in preserving the truthful VCG outcome (Dütting et al. 2015).

Recommended Literature

The results in this lecture can be found in the following two works:

- Hal R. Varian. Position Auctions. International Journal of Industrial Organization, Vol. 25: 1163–1178, 2007.

  (Model, definition of a symmetric Nash equilibrium, most of the results)

- Benjamin Edelman, Michael Ostrovsky, Michael Schwarz. Internet Advertising and the Generalized Second Price Auction: Selling Billions of Dollars Worth of Keywords. American Economic Review, Vol.97(1):242–259, 2007.

  (Model, similar concept of a locally envy-free equilibrium)

If you are curious about the bounds on the Price of Anarchy:

- Cragiannis et al. Bounding the Inefficiency of Outcomes in Generalized Second Price Auctions. Journal of Economic Theory, Vol. 156: 343–388, 2015.

  (Price of Anarchy bounds for GSP)

And a nice connection with the previous lectures on best-response mechanisms:

- Nisan et al. Best-response auctions. In EC, pp. 351-360, 2011.

  (GSP as a best-response mechanism – convergence and incentive compatibility)

The model in which α_j is different from a true quality β_j of the slots is studied here:

- Paul Dütting, Felix Fischer, David C. Parkes. Truthful Outcomes from Non-Truthful Position Auctions. Proc. of the 17th ACM Conference on Economics and Computation (EC), 2016

  (Increased robustness of non-truthful mechanisms)

A significant part of this notes is from prior year's notes by Paul Dütting available here:

- http://www.cadmo.ethz.ch/education/lectures/HS15/agt_HS2015/

Exercises
(during next exercise class - 17.12.2019)

We shall discuss and solve together this exercise.

Exercise 2. *This exercise about symmetric pure Nash equilibria (SNE). In particular, we want to prove this theorem stated in the lecture notes:*

> ***Theorem 10.*** *There exists always a symmetric pure Nash equilibrium whose revenue is the same as the revenue achieved by VCG on input the true valuations.*

Your task is to prove this theorem: for every valuations v, it is possible to construct bid vector b^{VCG} such that

$$P_s^{VCG}(v) = P_s^{GSP}(b^{VCG})$$

and b^{VCG} is a symmetric pure Nash equilibrium.
Hint: *it might be useful to use the simpler characterization of SPE in the lecture notes.*

Exercise 3. *Reconsider and compare **Theorem 10** with **Theorem 12** in the lecture notes, and check why we need **Theorem 10**. How do we know that SNE actually exist? Does **Theorem 10** also say something else about the revenue, which we cannot deduce from **Theorem 12**?*