Dual Graph

Prove that a set of edges in a connected plane graph $G$ forms a spanning tree of $G$ if and only if the duals of the remaining edges form a spanning tree of the dual graph $G^\ast$.

Petersen Graph

Let $G$ be an $n$-vertex (simple) planar graph with girth $k$ (the girth is the length of the shortest cycle of a graph). Prove that $G$ has at most $(n - 2)k/(k - 2)$ edges. Use this to prove that the Petersen graph (see Figure 1) is nonplanar.

![Figure 1: Petersen Graph](image-url)
Coloring Outerplanar Graphs

A graph is called outerplanar if it has a planar embedding in the plane such that all vertices lie on the outer (unbounded) face. Without using the Four Color Theorem, prove that every outerplanar graph is 3-colorable. Apply this to prove the Art Gallery Theorem: If an art gallery is laid out as a simple polygon with \( n \) sides (see Figure 2), then it is possible to place \( \lfloor n/3 \rfloor \) guards such that every point of the interior is visible to some guard. For \( n \geq 3 \), construct a polygon that requires \( \lceil n/3 \rceil \) guards.

Figure 2: Simple Polygon

Discussion of the exercises on 10.05.2007.