Min-Max Relations, Hall’s Theorem, and Matching-Algorithms

Graphs & Algorithms
Lecture 5
Min-Max Relations

• "A theorem stating equality between the answers to a minimization problem and a maximization problem over a class of instances" (D. West).

• Dual optimization problems:
  – maximization problem $\mathcal{M}$
  – minimization problem $\mathcal{N}$
  – $\forall$ feasible solutions $\mathcal{M} \in \mathcal{M}$ and $\mathcal{N} \in \mathcal{N}$: $\text{val}(\mathcal{M}) \leq \text{val}(\mathcal{N})$
  – If $\mathcal{M} = \mathcal{N}$, we have a proof of optimality!
  – min-max relations assert the existence of such short proofs

• Example: Menger’s theorem
Matchings in graphs

- **matching**: a set of independent edges, i.e., edges that share no endpoints
- **maximal matching**: a matching that cannot be extended by any other edge
- **maximum matching**: a matching of maximum cardinality among all maximal matchings
- A vertex is **matched** or **saturated** if any of its incident edges is in the matching
- **perfect matching** (**1-factor**): a matching that saturates all vertices
- **k-factor**: k-regular spanning subgraph
Theorem of König and Egerváry

- Let $G = (V, E)$ be an (undirected) graph.
- A vertex cover $C \subseteq V$ is a set of vertices such that, for all $e \in E$, we have $e \cap C \neq \emptyset$.

**Theorem** (König [1931], Egerváry [1931])
If $G = (A \cup B, E)$ is a bipartite graph, then the maximum size of a matching in $G$ equals the minimum size of a vertex cover.

**Proof**
Apply Menger’s theorem to $A$ and $B$. 


Menger’s Theorem

**Theorem** (multiple sources and sinks)
Let $G = (V, E)$ be a graph and $S, T \subseteq V$. Let
- $X \subseteq V$ be a set separating $S$ from $T$ of minimal size,
- $\mathcal{P}$ be a set of disjoint $S – T$ paths of maximal size.

Then we have $|X| = |\mathcal{P}|$.

**Corollary**
Let $G = (A \cup B, E)$ be a graph bipartite graph. Let
- $X \subseteq A \cup B$ be a set separating $A$ from $B$ of minimal size,
- $\mathcal{P}$ be a set of disjoint $A – B$ paths of maximal size.

Then we have $|X| = |\mathcal{P}|$. 
The Marriage Problem

• Given two groups, one of girls $G$ and one of boys $B$.
• Each girl $g \in G$ knows a some boys $\Gamma(g) \subseteq B$.
• Can all girls be married off to a boy they know?
• Obvious necessary condition: each subset $G' \subseteq G$ must satisfy
  $$|G'| \leq |\Gamma(G')|.$$  
• Is this also sufficient?
Hall's Theorem

**Theorem** (Hall, 1935)

A bipartite graph $G = (A \cup B, E)$ has a matching that saturates every vertex in $A$ if and only if for each $A' \subseteq A$, we have

$$|A'| \leq |\Gamma(A')|.$$ 

**Corollary** (Frobenius, 1917)

For all $k > 0$, every $k$-regular bipartite graph has a perfect matching.
Augmenting paths in bipartite graphs

• Let $G = (A \cup B, E)$ be a bipartite graph and $M$ be a fixed matching in $G$.

• A path $P = a \ldots b$ is called $M$-augmenting if
  – $a$ is some unsaturated vertex in $A$,
  – $b$ is some unsaturated vertex in $B$,
  – $P$ alternates between edges in $M$ and $E \setminus M$.

• If $P$ is an $M$-augmenting path, then $M \Delta P$ is a matching of size $|M| + 1$.

• Maximum bipartite matching algorithm:
  start with $M = \emptyset$ and extend this as long as there is an $M$-augmenting path (Exercise).

• Running time $O(|V| \cdot |E|)$
Computation of augmenting paths

**AugmentingPath**\((G = (A \cup B, E), M)\)

1. \(S \leftarrow \{a \in A : a \text{ is unsaturated by } M\}\)
2. for each vertex \(u \in A \cup B\)
   3. do \(\text{color}[u] \leftarrow \text{WHITE}\)
   4. \(\pi[u] \leftarrow \text{NIL}\)
5. \(Q \leftarrow \emptyset\)
6. for each vertex \(s \in S\)
   7. do \(\text{ENQUEUE}(Q, s)\)
   8. \(\text{color}[s] \leftarrow \text{GRAY}\)
9. while \(Q \neq \emptyset\)
10. do \(a \leftarrow \text{DEQUEUE}(Q)\)
11. for each \(b \in \text{Adj}[u]\)
12. do if \(\{a, b\} \notin M \land \text{color}[b] = \text{WHITE}\)
13. then \(\pi[b] \leftarrow a\)
14. if \(b\) is unsaturated by \(M\)
15. then return \(b\)
16. else \(a' \leftarrow \text{neighbor of } b\) in the matching
17. \(\pi[a'] \leftarrow b\)
18. \(\text{ENQUEUE}(Q, a')\)
19. \(\text{color}[a'] \leftarrow \text{GRAY}\)
20. \(\text{color}[b] \leftarrow \text{GRAY}\)
21. return \(\text{NIL}\)
Algorithm of Hopcroft and Karp

```
HOPCROFT-KARP(G = (A U B, E))
1  M ← ∅
2  repeat
3      let P = {P_1, P_2, ..., P_k} be a maximal set of vertex-disjoint
4          shortest M-augmenting paths
5      M ← MΔ(P_1 U P_2 U ... U P_k)
6  until P = ∅
7  return M
```

Theorem

The breadth-first phased maximum matching algorithm runs in $O(n^{1/2} \cdot m)$ time on bipartite graphs with $n$ vertices and $m$ edges.
Proof of the Hopcroft-Karp Algorithm

Lemma 1
If $M$ is a matching of size $r$ and $M^*$ is a matching of size $s > r$, then there exist at least $s - r$ vertex-disjoint $M$-augmenting paths. At least this many such paths can be found in $M \Delta M^*$. (Exercise)

Lemma 2
If $P$ is a shortest $M$-augmenting path and $P'$ is $M \Delta P$-augmenting, then we have
\[ |P'| \geq |P| + 2|P \cap P'| . \]

Lemma 3
If $P_1, P_2, ...$ is a list of successive shortest augmentations, then the augmentations of the same length are vertex disjoint paths.

Here paths are simply edge-sets.