Planar Graphs: Euler's Formula and Coloring

Graphs & Algorithms

Lecture 7
Jordan Curves

- A curve is a subset of $\mathbb{R}^2$ of the form
  \[ \alpha = \{ \gamma(x) : x \in [0, 1]\} , \]
  where $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ is a continuous mapping from
  the close interval $[0, 1]$ to the plane. $\gamma(0)$ and $\gamma(1)$ are
  called the endpoints of curve $\alpha$.

- A curve is closed if its first and last points are the same. A curve is simple if it has no repeated points except possibly first = last. A closed simple curve is called a Jordan-curve.

- Examples:
  - line segment between $p, q \in \mathbb{R}^2$: $x \mapsto xp + (1 - x)q$
  - circular arcs, Bezier curves without self-intersection, etc.
Drawing of Graphs

• A **drawing** of a (multi)graph $G$ is a function $f$ defined on $V(G) \cup E(G)$ that assigns
  - a point $f(v) \in \mathbb{R}^2$ to each vertex $v$ and
  - an $f(u), f(v)$-curve to each edge $\{u, v\}$, such that the images of vertices are distinct. A point in $f(e) \cap f(e')$ that is not a common endpoint is a **crossing**.

• A (multi)graph is **planar** if it has a drawing without crossings. Such a drawing is a **planar embedding** of $G$. A planar (multi)graph together with a particular planar embedding is called a **plane (multi)graph**.
Example: plane graph

drawing

plane embedding
Non-planar graphs

Proposition

$K_5$ and $K_{3,3}$ cannot be drawn without crossing.

Proof

Define the conflict graph of edges.

Jordan Curve Theorem (the unconscious ingredient)

A simple closed curve $C$ partitions the plane into exactly two faces, each having $C$ as a boundary.
Example: closed curve
Regions and faces

• An **open set** in the plane is a set $\mathcal{U} \subseteq \mathbb{R}^2$ such that for every $p \in \mathcal{U}$, all points within some small distance belong to $\mathcal{U}$.

• A **region** is an open set $\mathcal{U}$ that contains a $u, v$-curve for every pair $u, v \in \mathcal{U}$.

• The **faces** of a plane (multi)graph are the maximal regions of the plane that contain no points used in the embedding.

• A finite plane (multi)graph $G$ has one **unbounded face** (also called **outer face**).
Example: faces of a plane graph
Dual graph

• Denote the set of faces of a plane (multi)graph G by $F(G)$ and let $E(G) = \{e_1, \ldots, e_m\}$. Define the dual (multi-)graph $G^*$ of G by
  
  - $V(G^*) := F(G)$
  - $E(G^*) := \{e^*_1, \ldots, e^*_m\}$, where the endpoints of $e^*_i$ are the two (not necessarily distinct) faces $f', f'' \in F(G)$ on the two sides of $e_i$.

• Multiple edges and loops can appear in the dual of simple graphs.

• Different planar embeddings of the same planar graph could produce different duals.
Two observations about plane graphs

Proposition

Let \( l(F_i) \) denote the length of face \( F_i \) in a plane (multi)graph \( G \). Then we have

\[
2e(G) = \sum l(F_i).
\]

Proposition

Edges \( e_1, \ldots, e_r \in E(G) \) form a cycle in \( G \) if and only if \( e^*_1, \ldots, e^*_r \in E(G^*) \) form a minimal nonempty edge-cut in \( G^* \).
Euler's Formula

**Theorem** (Euler 1758)
If a connected plane multigraph $G$ has $n$ vertices, $e$ edges, and $f$ faces, then we have
$$f = e - n + 2.$$  

**Proof**
Induction on $n$ (for example).

**Corollary**
If $G$ is a plane multigraph with $k$ components, then
$$f = e - n + k + 1.$$  

**Remark**
Each embedding of a planar graph has the same number of faces.
Number of edges in planar graphs

Theorem
Let $G$ be a simple planar graph with $n \geq 3$ vertices and $e$ edges. Then we have $e \leq 3n - 6$. If $G$ is also triangle-free, then $e \leq 2n - 4$.

Corollary
$K_5$ and $K_{3,3}$ are not planar.

Proposition
For a simple plane graph $G$ on $n$ vertices, the following are equivalent:
- $G$ has $3n - 6$ edges
- $G$ is a triangulation (every face is a triangle)
- $G$ is a maximal planar graph (we cannot add edges)
Coloring maps
Coloring maps with 5 colors

Five Color Theorem (Heawood, 1890)
If $G$ is planar, then $\chi(G) \leq 5$.

Proof
• There is a vertex $v$ of degree at most 5.
• Modify a proper 5-coloring of $G - v$ so as to obtain a proper 5-coloring of $G$. A contradiction.
• Idea of modification: Kempe chains
Coloring maps with 4 colors

Four Color Theorem (Appel, Haken 1976)

If G is planar, then $\chi(G) \leq 4$.

Idea of the proof

- W.l.o.g. we can assume G is a planar triangulation.
- A configuration is a separating cycle $C \subseteq G$ (the ring) together with the portion of G inside C.
- For the Four Color Problem, a set of configurations is an unavoidable set if a minimum counterexample must contain some member of it.
- A configuration is reducible if a planar graph containing cannot be a minimal counterexample.
- Find an unavoidable set in which each configuration is reducible.
A problem with a long history

• Kempe's original attempt (1879) with a set of 3 unavoidable configurations was wrong.
• Appel and Haken, working with Koch, (1976) came up with a set of 1936 unavoidable configurations, each of which is reducible (1000 hours of computer time).
• Robertson, Sanders, Seymour and Thomas (1996) used a set of 633 unavoidable configurations (3 hours of computer time).