Exercise 1 (Greedy Coloring)

(a) Prove that there is an ordering \( v_1, v_2, \ldots, v_n \) of the vertices of \( G \) such that the greedy coloring algorithm yields an optimal coloring when using this ordering.

Let \( c : V(G) \rightarrow [k] \) with \( k = \chi(G) \) be an optimal coloring of the vertices of \( G \). Consider the color classes \( C_i = \{ v \in V(G) | c(v) = i \} \) and let \( k_i = |C_1 \cup C_2 \cup \ldots \cup C_i| \) for \( 1 \leq i \leq k \). Set \( k_0 = 0 \). Now, order the vertices of \( G \) such that for \( 1 \leq i \leq k \) we have \( C_i = \{ v_{k_i-1+1}, v_{k_i-1+2}, \ldots, v_{k_i} \} \), i.e. the ordering is built of blocks of consecutive vertices belonging to one color class. Given this ordering the greedy algorithm will color the graph with \( \chi(G) \) colors.

(b) Show that there is a tree \( T = (V, E) \) on \( n \) vertices and a permutation \( \pi : \{1, 2, \ldots, n\} \rightarrow V \) such that the algorithm Greedy-Coloring(\( T, \pi \)) needs \( \Omega(\log n) \) colors.

We construct an infinite family of rooted trees \( T_k (k \geq 0) \) as follows: Let \( T_0 \) be an isolated vertex and \( T_k \) the tree that is obtained by connecting a new root vertex \( v \) to the root vertices of copies of all trees \( T_0, T_1, \ldots, T_{k-1} \). Clearly we have \( n(T_k) = 2^k \). We want to show that there is an ordering \( \pi \) of the vertices of \( T_k \) such that Greedy-Coloring(\( T_k, \pi \)) = \( k+1 = \Omega(\log n) \). Such an ordering \( \pi \) can be defined recursively. \( T_0 \) has only one vertex and there is only one possible ordering. The ordering of the vertices of \( T_k \) is defined as follows: First visit the vertices of \( T_0 \), then those of \( T_1 \) etc. up to \( T_{k-1} \), and finally visit the new root vertex \( v \). (the vertices in each subtree are visited in the order defined for this subtree). It is easy to see by induction that using this ordering the greedy coloring algorithm will always need an additional \((k + 1)\)-th color to color the new root vertex \( v \).

Exercise 2 (Completing the proof of Brooks’ Theorem)

(For the whole proof: The proof presented on http://www.win.tue.nl/~wscor/OW/CO1b/brooks.pdf is similar to the one given in class and the tutorial.)

From the lecture, we know that we can assume that \( \Delta(G) \geq 3 \) and that \( G \) is \( \Delta(G) \)-regular and \( \kappa(G) = 2 \), i.e. \( G \) is 2-connected but not 3-connected.

Let \( \{x, y\} \) be a minimal cut-set and let \( C \) be one of the connected components of \( G \setminus \{x, y\} \). Then \( V(G) \setminus c \) is not-empty and we can consider the two graphs

\[ G_1 = G[C \cup \{x, y\}] \quad \text{and} \quad G_2 = G[V(G) \setminus C]. \]

We distinguish two cases:

Case 1: \( \{x, y\} \in E(G) \)

\( x \) and \( y \) both have neighbors in both \( G_1 \) and \( G_2 \). Hence, both \( G_1 \) and \( G_2 \) are not regular and by the part of the proof treated in the lecture, we know that they can be colored with \( \Delta(G) \) colors.
Let the resulting colorings be \( c_1 \) and \( c_2 \). Because \( \{x, y\} \in E(G) \), \( x \) and \( y \) have to get different colors in both colorings. We can hence swap colors to get a coloring for \( G \):

\[
c(v) = \begin{cases} 
  c_1(v) & \text{if } v \in V(G_1), \\
  c_2(v) & \text{if } v \in V(G_2), c_2(v) \notin \{a, b, c, d\} \\
  a & \text{if } v \in V(G_2), c_2(v) = c \\
  b & \text{if } v \in V(G_2), c_2(v) = d \\
  c & \text{if } v \in V(G_2), c_2(v) = a \\
  d & \text{if } v \in V(G_2), c_2(v) = b.
\end{cases}
\]

where we assume \( c_1(x) = a, c_1(y) = b, c_2(x) = c, c_2(y) = d \).

**Case 2:** \( \{x, y\} \notin E(G) \)

In this case, we add \( \{x, y\} \) to our three graphs, obtaining the graphs \( G', G'_1 \) and \( G'_2 \). If one of \( x \) and \( y \) has at least two neighbors in both \( G_1 \) and \( G_2 \), it has degree \( \leq \Delta(G) - 1 \) in both \( G'_1 \) and \( G'_2 \). In this case we can proceed as above. If e.g. \( \text{deg}_{G_1}(x) = \text{deg}_{G_2}(y) = 1 \), let \( x' \) and \( y' \) be their neighbors in \( G_1 \). Then \( \{x, y'\} \) is also a cut-set and it fulfills the condition of the first argument of this case.