Randomized Algorithms and Probabilistic Methods: Advanced Topics

Exercise 1
Recall that for a graph $G = (V, E)$ the edge expansion of $G$ is defined as
\[ h(G) = \min_{S \subseteq V, |S| \leq \frac{|V|}{2}} \frac{E(S, \bar{S})}{|S|}. \]
Show that for every $0 < \delta \leq 1$ the edge expansion of the random graph $G_{n,p}$ with $p \in \omega(\log n/n)$ a.a.s. satisfies
\[ h(G_{n,p}) \geq (1 - \delta)np/2. \]

Exercise 2
Consider the graph $G = (V, E)$ where the vertices can be partitioned into $k \geq 2$ sets of size $n \geq 1$, this is $V = V_1 \cup \ldots \cup V_k$ with $|V_i| = n$ for $1 \leq i \leq k$, and the edges are $E = \{\{u, v\} \mid u \in V_i \land v \in V_{i+1} \text{ with } 1 \leq i < k\} \cup \{\{u, u\} \mid u \in V_1 \lor u \in V_k\}$. Consider the following random walk on $G$: if the random walker is in a vertex in $V_2 \cup \ldots \cup V_{k-1}$, then he chooses its destination among all neighbors uniformely at random and if he is in $V_1 \cup V_k$, then he does not move with probability $1/2$ and otherwise chooses its destination among all neighbors uniformly at random. Compute the stationary distribution $\pi$, show that the random walk converges to $\pi$ and show
\[ t_{mix} \leq O(k^2 \log (nk)) \]
using the flow method.

Exercise 3
Compute the eigenvalues of the hypercube.

Exercise 4
Let $G$ be a (undirected) graph. Consider the random walk on $G$ and let $P$ be the corresponding transition matrix.

(a) Show that $G$ has a bipartite component iff $-1$ is an eigenvalue of $P$.

(b) Show that the number of connected components of $G$ is equal to the multiplicity of the eigenvalue $1$ of $P$.

You can use the following theorem:

**Theorem 1** (Perron-Frobenius). Any irreducible, stochastic matrix $P$ has eigenvalue $1$ with multiplicity $1$ and the corresponding eigenvector is positive.